Appendices

Appendix A2 pp. (574–576)

Exercises

1. Step 1: The formula holds for $n = 1$, because $|x_1| = |x_1|$.
Step 2: Suppose $|x_1 + x_2 + \cdots + x_k| \le |x_1| + |x_2| + \cdots + |x_k|$.
Then $|x_1 + x_2 + \cdots + x_{k+1}| \le |x_1 + x_2 + \cdots + x_k| + |x_{k+1}|$ by the triangle equality. So, by the transitivity of $\le$,
$|x_1 + x_2 + \cdots + x_{k+1}| \le |x_1| + |x_2| + \cdots + |x_{k+1}|$.
The mathematical induction principle now guarantees the original formula for all n.

2. Step 1: The formula holds for $n = 1$, because
$$1 + r = \frac{(1+r)(1-r)}{1-r} = \frac{1-r^2}{1-r}.$$
Step 2: Suppose
$1 + r + r^2 + \cdots + r^k = \dfrac{1-r^{k+1}}{1-r}$. Then
$$\begin{aligned}1 + r + r^2 + \cdots + r^{k+1} &= \frac{1-r^{k+1}}{1-r} + r^{k+1}\\ &= \frac{1-r^{k+1} + r^{k+1}(1-r)}{1-r}\\ &= \frac{1-r^{k+2}}{1-r}.\end{aligned}$$
The mathematical induction principle now guarantees the original formula for every positive integer n.

3. Step 1: The formula holds for $n = 1$, because $\dfrac{d}{dx}(x) = 1$.

Step 2: Suppose $\dfrac{d}{dx}(x^k) = kx^{k-1}$. Then
$$\begin{aligned}\frac{d}{dx}(x^{k+1}) &= \frac{d}{dx}(x \cdot x^k)\\ &= x \cdot \frac{d}{dx}(x^k) + x^k \cdot \frac{d}{dx}(x)\\ &= x \cdot kx^{k-1} + x^k\\ &= kx^k + x^k\\ &= (k+1)x^k.\end{aligned}$$
The mathematical induction principle now guarantees the original formula for any positive integer n.

4. Step 1: The formula holds for $n = 1$, because $f(x_1) = f(x_1)$.
Step 2: Suppose $f(x_1 x_2 \cdots x_k) = f(x_1) + f(x_2) + \cdots + f(x_k)$.
Then by the given property,
$$\begin{aligned}&f(x_1 x_2 \cdots x_{k+1})\\ &= f(x_1 x_2 \cdots x_k) + f(x_{k+1})\\ &= f(x_1) + f(x_2) + \cdots + f(x_{k+1}).\end{aligned}$$
The mathematical induction principle now guarantees the original formula for every positive integer n.

5. Step 1: The formula holds for $n = 1$, because $\dfrac{2}{3} = 1 - \dfrac{1}{3}$.
Step 2:
Suppose $\dfrac{2}{3^1} + \dfrac{2}{3^2} + \cdots + \dfrac{2}{3^k} = 1 - \dfrac{1}{3^k}$. Then
$$\begin{aligned}\frac{2}{3^1} + \frac{2}{3^2} + \cdots + \frac{2}{3^{k+1}} &= 1 - \frac{1}{3^k} + \frac{2}{3^{k+1}}\\ &= 1 - \frac{3-2}{3^{k+1}}\\ &= 1 - \frac{1}{3^{k+1}}.\end{aligned}$$
The mathematical induction principle now guarantees the original formula for all positive integers n.

6. Experiment:

n	1	2	3	4	5	6	7
$n!$	1	2	6	24	120	720	5040
n^3	1	8	27	64	125	216	343

Step 1: The inequality holds for $n = 6$, because $720 > 216$.
Step 2: Suppose $k! > k^3$. Then $(k+1)! > (k+1)k^3$.
For $k \ge 4$, $\quad k > 1 + \dfrac{2}{k} + \dfrac{1}{k^2}$ $\left(\text{since } \dfrac{2}{k} < 2 \text{ and } \dfrac{1}{k^2} < 1\right)$, and so $k^3 > k^2 + 2k + 1 = (k+1)^2$.
So $(k+1)k^3 > (k+1)^3$, and thus by transitivity of $>$, $(k+1)! > (k+1)^3$.
The mathematical induction principle now guarantees the original inequality for all $n \ge 6$.

7. Experiment:

n	1	2	3	4	5	6
2^n	2	4	8	16	32	64
n^2	1	4	9	16	25	36

Step 1: The inequality holds for $n = 5$, because $32 > 25$.

Step 2: Suppose $2^k > k^2$. For $k \geq 3$, $k > 2 + \frac{1}{k}$ $\left(\text{since } \frac{1}{k} < 1\right)$, and so $k^2 > 2k + 1$. Then by the transitivity of $>$, $2^k > 2k + 1$. And thus $2^{k+1} = 2^k + 2^k > k^2 + 2k + 1 = (k+1)^2$.

The mathematical induction principle now guarantees the original inequality for all $n \geq 5$.

8. Step 1: The inequality holds for $n = -3$, because $2^{-3} = \frac{1}{8}$.

Step 2: Suppose $2^k \geq \frac{1}{8}$. Then $2^{k+1} = 2 \cdot 2^k \geq \frac{2}{8} = \frac{1}{4}$, and by the transitivity of $\geq$ and the fact that $\frac{1}{4} \geq \frac{1}{8}$, $2^{k+1} \geq \frac{1}{8}$.

The mathematical induction principle now guarantees the original inequality for $n \geq -3$.

9. Step 1: The formula holds for $n = 1$, because

$$1^2 = \frac{1\left(1+\frac{1}{2}\right)(1+1)}{3} = 1.$$

Step 2: Suppose

$$1^2 + 2^2 + \cdots + k^2 = \frac{k\left(k+\frac{1}{2}\right)(k+1)}{3}.$$

Then $1^2 + 2^2 + \cdots + (k+1)^2$

$$= \frac{k\left(k+\frac{1}{2}\right)(k+1)}{3} + (k+1)^2$$

$$= \frac{k\left(k+\frac{1}{2}\right)(k+1) + 3(k+1)^2}{3}$$

$$= \frac{k^3 + \left(\frac{9}{2}\right)k^2 + \left(\frac{13}{2}\right)k + 3}{3}$$

$$= \frac{(k+1)\left(k+\frac{3}{2}\right)(k+2)}{3}$$

$$= \frac{(k+1)\left[(k+1)+\frac{1}{2}\right][(k+1)+1]}{3}.$$

The mathematical induction principle now guarantees the original formula for all positive integers n.

10. Step 1: The formula holds for $n = 1$, because

$$1^3 = \left(\frac{1(1+1)}{2}\right)^2 = 1.$$

Step 2: Suppose

$$1^3 + 2^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2}\right)^2.$$

Then

$1^3 + 2^3 + \cdots + (k+1)^3$

$$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)(k+1)^2$$

$$= \left(\frac{k^2}{4} + k + 1\right)(k+1)^2$$

$$= \left[\frac{(k+2)^2}{4}\right](k+1)^2$$

$$= \left(\frac{(k+1)(k+2)}{2}\right)^2.$$

The mathematical induction principle now guarantees the original formula for all positive integers n.

11. **(a)** Step 1: The formula holds for $n = 1$, because

$$\sum_{k=1}^{1}(a_k + b_k) = \sum_{k=1}^{1} a_k + \sum_{k=1}^{1} b_k = a_1 + b_1.$$

Step 2: Suppose

$$\sum_{k=1}^{i}(a_k + b_k) = \sum_{k=1}^{i} a_k + \sum_{k=1}^{i} b_k. \text{ Then}$$

$$\sum_{k=1}^{i+1}(a_k + b_k)$$
$$= \left[\sum_{k=1}^{i}(a_k + b_k)\right] + (a_{i+1} + b_{i+1})$$
$$= \left[\sum_{k=1}^{i} a_k\right] + \left[\sum_{k=1}^{i} b_k\right] + a_{i+1} + b_{i+1}$$
$$= \sum_{k=1}^{i+1} a_k + \sum_{k=1}^{i+1} b_k.$$

The mathematical induction principle now guarantees the original formula for every positive integer n.

(b) Step 1: The formula holds for $n = 1$, because

$$\sum_{k=1}^{1}(a_k - b_k) = \sum_{k=1}^{1} a_k - \sum_{k=1}^{1} b_k = a_1 - b_1.$$

Step 2: Suppose

$$\sum_{k=1}^{i}(a_k - b_k) = \sum_{k=1}^{i} a_k - \sum_{k=1}^{i} b_k. \text{ Then}$$

$$\sum_{k=1}^{i+1}(a_k - b_k)$$
$$= \left[\sum_{k=1}^{i}(a_k - b_k)\right] + (a_{i+1} - b_{i+1})$$
$$= \left[\sum_{k=1}^{i} a_k\right] - \left[\sum_{k=1}^{i} b_k\right] + a_{i+1} - b_{i+1}$$
$$= \sum_{k=1}^{i+1} a_k - \sum_{k=1}^{i+1} b_k.$$

The mathematical induction principle now guarantees the original formula for every positive integer n.

(c) Step 1: The formula holds for $n = 1$, because

$$\sum_{k=1}^{1} ca_k = c \cdot \sum_{k=1}^{1} a_k = ca_1.$$

Step 2: Suppose $\sum_{k=1}^{i} ca_k = c \cdot \sum_{k=1}^{i} a_k$. Then

$$\sum_{k=1}^{i+1} ca_{k+1} = \left[\sum_{k=1}^{i} ca_k\right] + ca_{k+1}$$
$$= \left[c \cdot \sum_{k=1}^{i} a_k\right] + ca_{k+1}$$
$$= c\left[\left(\sum_{k=1}^{i} a_k\right) + a_{k+1}\right]$$
$$= c \cdot \sum_{k=1}^{i+1} a_{k+1}.$$

The mathematical induction principle how guarantees the original formula for every positive integer n.

(d) Step 1: The formula $\sum_{k=1}^{n} c = n \cdot c$ holds for $n = 1$, because $\sum_{k=1}^{1} c = 1 \cdot c = c$.

Step 2: Suppose $\sum_{k=1}^{i} c = i \cdot c$. Then

$$\sum_{k=1}^{i+1} c = i \cdot c + c = (i+1) \cdot c.$$

The mathematical induction principle now guarantees the original formula for every positive integer n.

12. Step 1: The formula holds for $n = 1$ (and every real number x), because

$$\left|x^1\right| = |x|^1 = |x|.$$

Step 2: Suppose $\left|x^k\right| = |x|^k$. Then

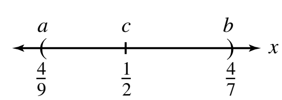
$$\left|x^{k+1}\right| = \left|x^k \cdot x\right| = \left|x^k\right| \cdot |x| = |x|^k \cdot |x| = |x|^{k+1}.$$

The mathematical induction principle now guarantees the original formula for every positive integer n (and every real number x).

Appendix A3 (pp. 577–584)

Exercises

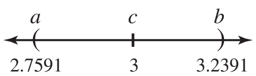
1.



Step 1:

$$\left|x - \frac{1}{2}\right| < \delta \Rightarrow -\delta < x - \frac{1}{2} < \delta$$
$$\Rightarrow -\delta + \frac{1}{2} < x < \delta + \frac{1}{2}$$

Step 2:
$-\delta+\frac{1}{2}=\frac{4}{9}\Rightarrow\delta=\frac{1}{18}$, or
$\delta+\frac{1}{2}=\frac{4}{7}\Rightarrow\delta=\frac{1}{14}$.
The value of δ which assures
$\left|x-\frac{1}{2}\right|<\delta\Rightarrow\frac{4}{9}<x<\frac{4}{7}$ is the smaller values,
$\delta=\frac{1}{18}$.

2. 

Step 1:
$|x-3|<\delta\Rightarrow-\delta<x-3<\delta$
$\Rightarrow-\delta+3<x<\delta+3$
Step 2:
$-\delta+3=2.7591\Rightarrow\delta=0.2409$, or
$\delta+3=3.2391\Rightarrow\delta=0.2391$.
The value of δ which assures
$|x-3|<\delta\Rightarrow2.7591<x<3.2391$ is the smaller value, $\delta=0.2391$.

3. Step 1:
$|x-3|<\delta\Rightarrow-\delta<x-3<\delta$
$\Rightarrow-\delta+3<x<\delta+3$
Step 2: From the graph,
$-\delta+3=2.61\Rightarrow\delta=0.39$, or
$\delta+3=3.41\Rightarrow\delta=0.41$; thus $\delta=0.39$.

4. Step 1:
$|x-(-1)|<\delta\Rightarrow-\delta<x+1<\delta$
$\Rightarrow-\delta-1<x<\delta-1$
Step 2: From the graph,
$-\delta-1=-\frac{16}{9}\Rightarrow\delta=\frac{7}{9}\approx0.77$, or
$\delta-1=-\frac{16}{25}\Rightarrow\frac{9}{25}=0.36$; thus
$\delta=\frac{9}{25}=0.36$.

5. Step 1:
$|(2x-2)-(-6)|<0.02\Rightarrow|2x+4|<0.02$
$\Rightarrow-0.02<2x+4<0.02$
$\Rightarrow-4.02<2x<-3.98\Rightarrow-2.01<x<-1.99$
Step 2:
$|x-(-2)|<\delta\Rightarrow-\delta<x+2<\delta$
$\Rightarrow-\delta-2<x<\delta-2\Rightarrow\delta=0.01$.

6. Step 1:
$\left|\sqrt{x+1}-1\right|<0.1\Rightarrow-0.1<\sqrt{x+1}-1<0.1$
$\Rightarrow0.9<\sqrt{x+1}<1.1\Rightarrow0.81<x+1<1.21$
$\Rightarrow-0.19<x<0.21$
Step 2: $|x-0|<\delta\Rightarrow-\delta<x<\delta\Rightarrow\delta=0.19$.

7. Step 1:
$\left|\sqrt{19-x}-3\right|<1\Rightarrow-1<\sqrt{19-x}-3<1$
$\Rightarrow2<\sqrt{19-x}<4\Rightarrow4<19-x<16$
$\Rightarrow-4>x-19>-16\Rightarrow15>x>3$
or $3<x<15$
Step 2:
$|x-10|<\delta\Rightarrow-\delta<x-10<\delta$
$\Rightarrow-\delta+10<x<\delta+10$.
Then $-\delta+10=3\Rightarrow\delta=7$, or $\delta+10=15$
$\Rightarrow\delta=5$; thus $\delta=5$.

8. Step 1:
$\left|x^2-4\right|<0.5\Rightarrow-0.5<x^2-4<0.5$
$\Rightarrow3.5<x^2<4.5\Rightarrow\sqrt{3.5}<|x|<\sqrt{4.5}$
$\Rightarrow-\sqrt{4.5}<x<-\sqrt{3.5}$, for x near -2.
Step 2:
$|x-(-2)|<\delta\Rightarrow-\delta<x+2<\delta$
$\Rightarrow-\delta-2<x<\delta-2$.
Then $-\delta-2=-\sqrt{4.5}$
$\Rightarrow\delta=\sqrt{4.5}-2\approx0.1213$,
or $\delta-2=-\sqrt{3.5}=\delta=2-\sqrt{3.5}\approx0.1292$;
thus $\delta=\sqrt{4.5}-2\approx0.121$.

9. **(a)**
$$\lim_{x\to-5}\frac{x^2+6x+5}{x+5}=\lim_{x\to-5}\frac{(x+5)(x+1)}{x+5}=\lim_{x\to-5}(x+1)=-4,\ x\neq-5.$$

(b) Step 1:
$\left|\left(\frac{x^2+6x+5}{x+5}\right)-(-4)\right|<0.05$
$\Rightarrow-0.05<\frac{(x+5)(x+1)}{x+5}+4<0.05$
$\Rightarrow-4.05<x+1<-3.95,\ x\neq-5$
$\Rightarrow-5.05<x<-4.95,\ x\neq-5$.

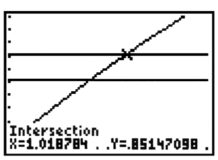
Step 2:
$|x-(-5)| < \delta \Rightarrow -\delta < x+5 < \delta \Rightarrow -\delta - 5 < x < \delta - 5.$
Then $-\delta - 5 = -5.05 \Rightarrow \delta = 0.05,$ or $\delta - 5 = -4.95 \Rightarrow \delta = 0.05;$ thus $\delta = 0.05.$

10. **(a)** $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (4-2x) = 2$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (6x-4) = 2,$ so $\lim_{x\to 1} f(x) = 2.$

(b) Step 1:
$x < 1$:
$(4-2x) - 2 < 0.5 \Rightarrow 2 - 2x < 0.5 \Rightarrow x \geq \frac{2-0.5}{2} = \frac{3}{4};$
$x \geq 1$:
$(6x-4) - 2 < 0.5 \Rightarrow 6x - 6 < 0.5 \Rightarrow x < \frac{6+0.5}{6} = \frac{13}{12}.$
Step 2:
$|x-1| < \delta \Rightarrow -\delta < x-1 < \delta \Rightarrow 1-\delta < x < 1+\delta.$
Then $1-\delta = \frac{3}{4} \Rightarrow \delta = \frac{1}{4},$ or $1+\delta = \frac{13}{12} \Rightarrow \delta = \frac{1}{12}.$ Choose $\delta = \frac{1}{12}.$

11. **(a)** $\lim_{x\to 1} (\sin x) = \sin 1 \approx 0.841$

(b) Step 1:
$|\sin x - \sin 1| < 0.01 \Rightarrow -0.01 < \sin x - \sin 1 < 0.01 \Rightarrow \sin 1 - 0.01 < \sin x < \sin 1 + 0.01$
$\Rightarrow \sin^{-1}(\sin 1 - 0.01) < x < \sin^{-1}(\sin 1 + 0.01)$
Step 2: $|x-1| < \delta \Rightarrow -\delta < x-1 < \delta \Rightarrow 1-\delta < x < 1+\delta.$
Then $1-\delta = \sin^{-1}(\sin 1 - 0.01) \Rightarrow \delta = 1 - \sin^{-1}(\sin 1 - 0.01) \approx 0.0182,$ or
$1+\delta = \sin^{-1}(\sin 1 + 0.01) \Rightarrow \delta = \sin^{-1}(\sin 1 + 0.01) - 1 \approx 0.0188.$
Choose $\delta = 0.018.$
Alternately, graph $y_1 = \sin x,$ $y_2 = \sin 1 - 1,$ and $y_3 = \sin 1 + 1.$ The curve intersects the lines at $x \approx 0.98175 = 1 - 0.01825$ and at $x \approx 1.01878 = 1 + 0.01878.$ We may choose $\delta = 0.018.$



[0.9, 1.1] by [0.78, 0.88]

12. **(a)** $\lim_{x\to -1} \frac{x}{x^2-4} = \frac{-1}{(-1)^2-4} = \frac{1}{3}$

(b) Step 1:

$$\left|\frac{x}{x^2-4}-\frac{1}{3}\right|<0.1$$

$$\Rightarrow -0.1<\frac{x}{x^2-4}-\frac{1}{3}<0.1$$

$$\Rightarrow \frac{7}{30}<\frac{x}{x^2-4}<\frac{13}{30}$$

$$\Rightarrow \frac{7}{30}x^2-\frac{28}{30}>x>\frac{13}{30}x^2-\frac{52}{30}$$

for x near -1. Then

$$\frac{7}{30}x^2-x-\frac{28}{30}>0>\frac{13}{30}x^2-x-\frac{52}{30},$$

which using the quadratic formula implies

$$x<\frac{15-\sqrt{421}}{7}\approx -0.7883 \text{ or}$$

$$x>\frac{15+\sqrt{421}}{7}\approx 5.0740, \text{ and also}$$

$$\frac{15-\sqrt{901}}{13}\approx -1.1551<x<\frac{15+\sqrt{901}}{13}\approx 3.4628.$$

Thus $\dfrac{15-\sqrt{901}}{13}<x<\dfrac{15-\sqrt{421}}{7}.$

Step 2:

$$|x-(-1)|<\delta \Rightarrow -\delta<x+1<\delta$$

$$\Rightarrow -\delta-1<x<\delta-1.$$

Then $-\delta-1=\dfrac{15-\sqrt{901}}{13}$

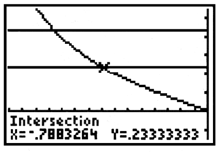
$$\Rightarrow \delta=\frac{\sqrt{901}-28}{13}\approx 0.1551, \text{ or}$$

$$\delta-1=\frac{15-\sqrt{421}}{7}$$

$$\Rightarrow \delta=\frac{22-\sqrt{421}}{7}\approx 0.2117.$$

Choose $\delta=0.155$.

Alternately, graph $y_1=\dfrac{x}{x^2-4}$, $y_2=\dfrac{1}{3}-0.1$, and $y_3=\dfrac{1}{3}+0.1$. The curve intersects the lines at $x\approx -1.15513=-1-0.15513$ and at $x\approx -0.78833=-1+0.21167$. We may choose $\delta=0.155$.



$[-1.5, 0]$ by $[-0.15, 0.55]$

13. Step 1:

For $x\neq 1$,

$$|x^2-1|<\varepsilon \Rightarrow -\varepsilon<x^2-1<\varepsilon$$

$$\Rightarrow 1-\varepsilon<x^2<1+\varepsilon$$

$$\Rightarrow \sqrt{1-\varepsilon}<|x|<\sqrt{1+\varepsilon}$$

$$\Rightarrow \sqrt{1-\varepsilon}<x<\sqrt{1+\varepsilon} \text{ near } x=1.$$

Step 2:

$$|x-1|<\delta \Rightarrow -\delta<x-1<\delta$$

$$\Rightarrow -\delta+1<x<\delta+1.$$

Then $-\delta+1=\sqrt{1-\epsilon}\Rightarrow \delta=1-\sqrt{1-\varepsilon}$,

or $\delta+1=\sqrt{1+\varepsilon}\Rightarrow \delta=\sqrt{1+\varepsilon}-1$.

Choose $\delta=\min\left\{1-\sqrt{1-\varepsilon}, \sqrt{1+\varepsilon}-1\right\}$, that is, the smaller of the two distances.

14. Step 1:

$$\left|\frac{1}{x^2}-\frac{1}{3}\right|<\varepsilon \Rightarrow -\varepsilon<\frac{1}{x^2}-\frac{1}{3}<\varepsilon$$

$$\Rightarrow \frac{1}{3}-\varepsilon<\frac{1}{x^2}<\frac{1}{3}+\varepsilon$$

$$\Rightarrow \frac{1-3\varepsilon}{3}<\frac{1}{x^2}<\frac{1+3\varepsilon}{3}$$

$$\Rightarrow \frac{3}{1-3\varepsilon}>x^2>\frac{3}{1+3\varepsilon}$$

$$\Rightarrow \sqrt{\frac{3}{1+3\varepsilon}}<|x|<\sqrt{\frac{3}{1-3\varepsilon}},$$

or $\sqrt{\dfrac{3}{1+3\varepsilon}}<x<\sqrt{\dfrac{3}{1-3\varepsilon}}$ for x near $\sqrt{3}$.

Step 2:

$$|x-\sqrt{3}|<\delta \Rightarrow -\delta<x-\sqrt{3}<\delta$$

$$\Rightarrow \sqrt{3}-\delta<x<\sqrt{3}+\delta.$$

Then $\sqrt{3}-\delta=\sqrt{\dfrac{3}{1+3\varepsilon}}\Rightarrow \delta=\sqrt{3}-\sqrt{\dfrac{3}{1+3\varepsilon}}$,

or $\sqrt{3}+\delta=\sqrt{\dfrac{3}{1-3\varepsilon}}\Rightarrow \delta=\sqrt{\dfrac{3}{1-3\varepsilon}}-\sqrt{3}$.

Choose

$$\delta=\min\left\{\sqrt{3}-\sqrt{\frac{3}{1+3\varepsilon}}, \sqrt{\frac{3}{1-3\varepsilon}}-\sqrt{3}\right\}.$$

15. **(a)** $\sqrt{(5+\delta)-5}=\epsilon \Rightarrow \sqrt{\delta}=\epsilon \Rightarrow \delta=\epsilon^2$

$$\Rightarrow I=(5, 5+\epsilon^2)$$

(b) $\displaystyle\lim_{x\to 5^+}\sqrt{x-5}=0$

16. **(a)** $\sqrt{4-(4-\delta)}=\varepsilon \Rightarrow \sqrt{\delta}=\varepsilon \Rightarrow \delta=\varepsilon^2$
$\Rightarrow I=(4-\varepsilon^2, 4)$

(b) $\lim_{x\to 4^-} \sqrt{4-x}=0$

17. If L, c, and k are real numbers and $\lim_{x\to c} f(x)=L$, show that for any $\varepsilon>0$, there is a $\delta>0$ such that
$0<|x-c|<\delta \Rightarrow |k\cdot f(x)=k\cdot L|<\varepsilon.$

Proof: For any $\varepsilon>0$, let $\varepsilon'=\frac{\varepsilon}{|k|}$. Since $\lim_{x\to c} f(x)+L$, there is a $\delta>0$ such that

$0<|x-c|<\delta \Rightarrow |f(x)-L|<\varepsilon'=\frac{\varepsilon}{|k|}$.

Therefore,
$0<|x-c|<\delta \Rightarrow |k\cdot f(x)-k\cdot L|<\varepsilon.$

18. If L, M, and c are real numbers and $\lim_{x\to c} f(x)=L$ and $\lim_{x\to c} g(x)=M$, show that for any $\varepsilon>0$, there is a $\delta>0$ such that
$0<|x-c|<\delta \Rightarrow |f(x)-g(x)-(L+M)|<\varepsilon.$
Proof:
$|f(x)-g(x)-(L-M)|$
$=|f(x)-L+M-g(x)|$
$\le|f(x)-L|+|M-g(x)|$
$=|f(x)-L|+|g(x)-M|$
by the triangle inequality. Since $\lim_{x\to c} f(x)=L$, $\lim_{x\to c} g(x)=M$, for any ε there exist δ_1, δ_2 such that

$0<|x-c|<\delta_1 \Rightarrow |f(x)-L|<\frac{\varepsilon}{2}$ and

$0<|x-c|<\delta_2 \Rightarrow |g(x)-M|<\frac{\varepsilon}{2}.$

Choose $\delta=\min\{\delta_1,\delta_2\}$. Then
$0<|x-c|<\delta$
$\Rightarrow |f(x)-L|+|g(x)-M|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$
$\Rightarrow |f(x)-g(x)-(L-M)|<\epsilon.$

19. $\lim_{x\to c}[f_1(x)+f_2(x)+f_3(x)\}$
$=\lim_{x\to c}[f_1(x)+f_2(x)]+L_3$
$=L_1+L_2+L_3$, by two applications of the Sum Rule.
To generalize:
Step 1 $(n=1)$: $\lim_{x\to c} f_1(x)=L_1$ as given.
Step 2: Suppose
$\lim_{x\to c}\left[f_1(x)+f_2(x)+\cdots+f_k(x)\right]$
$=L_1+L_2+\cdots+L_k$. Then
$\lim_{x\to c}\left[f_1(x)+f_2(x)+\cdots+f_{k+1}(x)\right]$
$=\lim\left[f_1(x)+f_2(x)+\cdots+f_k(x)\right]+L_{k+1}$
$=L_1+L_2+\cdots+L_{k+1}$, by the Sum Rule.

20. Step 1 $(n=1)$: $\lim_{x\to c} f_1(x)=L_1$, as given.
Step 2: Suppose $\lim_{x\to c}(f_1(x\cdot f_2(x)\cdot\cdots\cdot f_k(x))$
$=L_1\cdot L_2\cdot\cdots\cdot L_k$.
Then
$\lim_{x\to c}(f_1(x)\cdot f_2(x)\cdot\cdots\cdot f_{k+1}(x))$
$=\lim_{x\to c}(f_1(x))f_2(x)\cdot\cdots\cdot f_k(x))\cdot L_{k+1}$
$=L_1\cdot L_2\cdot\cdots\cdot L_{k+1}$, by the Product Rule.

21. $\lim_{x\to c} x^n=\lim_{x\to c}\underbrace{(x\cdot x\cdot\cdots\cdot x)}_{n\text{ factors}}=\underbrace{c\cdot c\cdot\cdots\cdot c}_{n\text{ factors}}=c^n$

22. $\lim_{x\to c} f(x) = \lim_{x\to c}(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)$

$= \lim_{x\to c} a_n x^n + \lim_{x\to c} a_{n-1}x^{n-1} + \cdots + \lim_{x\to c} a_1 x + \lim_{x\to c} a_0$

$= a_n \lim_{x\to c} x^n + a_{a-1} \lim_{x\to c} x^{n-1} + \cdots + a_1 \lim_{x\to c} x + \lim_{x\to c} a_0$

$= a_n c^n + a_{n-1}c^{n-1} + \cdots a_1 c + a_0 = f(c)$, where in addition to the items given in the problem, the Constant Multiple Rule was used (to move the coefficients out of the scope of the limit signs).

23. By the Quotient Rule, $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{f(c)}{g(c)}$.

24. From the continuity of g, for any $\epsilon > 0$ there is a $\delta > 0$ such that $0 < |f(x) - f(c)| < \delta \Rightarrow |g(f(x)) - g(f(c))| < \varepsilon$.

The second inequality also holds when $f(x) = f(c)$, so $|f(x) - f(c)| < \delta \Rightarrow |g(f(x)) - g(f(c))| < \varepsilon$. But from the continuity of f, there is a $\gamma > 0$ such that $0 < |x - c| < \gamma \Rightarrow |f(x) - f(c)| < \delta$. So for any $\varepsilon > 0$ there is a $\gamma > 0$ such that $0 < |x - c| < \gamma \Rightarrow |g(f(x)) - g(f(c))| < \varepsilon$, which means that $g(f(x)) = g \circ f$ is continuous at c.

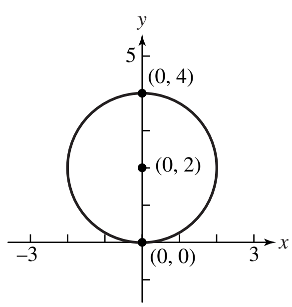
Appendix A5.1 (pp. 586–598)

Exercises

1. $(x-h)^2 + (y-k)^2 = a^2$

$(x-0)^2 + (y-2)^2 = 2^2$

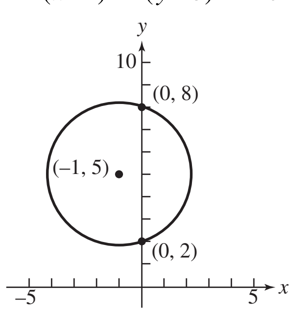
$x^2 + (y-2)^2 = 4$



2. $(x-h)^2 + (y-k)^2 = a^2$

$[x-(-1)]^2 + (y-5)^2 = \left(\sqrt{10}\right)^2$

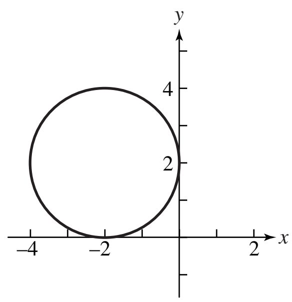
$(x+1)^2 + (y-5)^2 = 10$



3. Complete the squares.

$$x^2+y^2+4x-4y+4=0$$
$$x^2+4x+4+y^2-4y+4=4$$
$$(x+2)^2+(y-2)^2=2^2$$

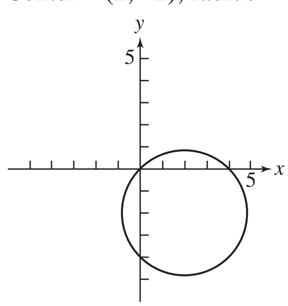
Center $=(-2, 2)$; radius $=2$



4. Complete the squares.

$$x^2+y^2-4x+4y=0$$
$$x^2-4x+4+y^2+4y+4=8$$
$$(x-2)^2+(y+2)^2=\left(2\sqrt{2}\right)^2$$

Center $=(2, -2)$; radius $=2\sqrt{2}$



5. The circle with center at (1, 0) and radius 2 plus its interior.

6. The region exterior to the unit circle and interior to the circle with center at (0, 0) and radius 2.

7. $y^2=8x \Rightarrow 4p=8 \Rightarrow p=2$; focus is (2, 0), directrix is $x=-2$

8. $y^2=-4x \Rightarrow 4p=4 \Rightarrow p=1$; focus is (–1, 0), directrix is $x=1$

9. $x^2=-6y \Rightarrow 4p=6 \Rightarrow p=\frac{3}{2}$; focus is $\left(0, -\frac{3}{2}\right)$, directrix is $y=\frac{3}{2}$

10. $x^2=2y \Rightarrow 4p=2 \Rightarrow p=\frac{1}{2}$; focus is $\left(0, \frac{1}{2}\right)$, directrix is $y=-\frac{1}{2}$

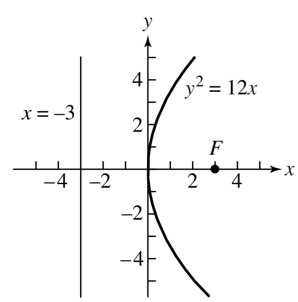
11. $\frac{x^2}{4}-\frac{y^2}{9}=1 \Rightarrow c=\sqrt{4+9}=\sqrt{13}$ $\Rightarrow$ foci are $\left(\pm\sqrt{13}, 0\right)$; vertices are $(\pm 2, 0)$; asymptotes are $y=\pm\frac{3}{2}x$

12. $\frac{x^2}{4}+\frac{y^2}{9}=1 \Rightarrow c=\sqrt{9-4}=\sqrt{5} \Rightarrow$ foci are $\left(0, \pm\sqrt{5}\right)$; vertices are $(0, \pm 3)$

13. $\frac{x^2}{2}+y^2=1 \Rightarrow c=\sqrt{2-1}=1 \Rightarrow$ foci are $(\pm 1, 0)$; vertices are $\left(\pm\sqrt{2}, 0\right)$

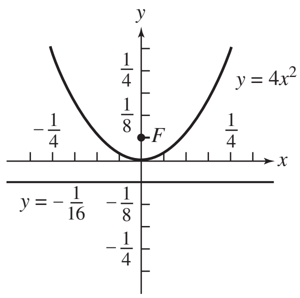
14. $\frac{y^2}{4}-x^2=1 \Rightarrow c=\sqrt{4+1}=\sqrt{5} \Rightarrow$ foci are $\left(0, \pm\sqrt{5}\right)$; vertices are $(0, \pm 2)$; asymptotes are $y=\pm 2x$

15. $y^2=12x \Rightarrow 4p=12 \Rightarrow p=3$; focus is (3, 0), directrix is $x=-3$



16. $y = 4x^2 \Rightarrow x^2 = \frac{1}{4}y \Rightarrow 4p = \frac{1}{4} \Rightarrow p = \frac{1}{16}$;

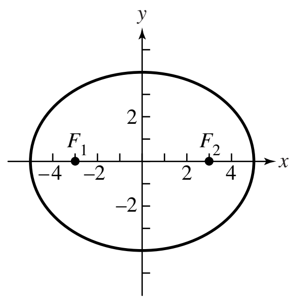
focus is $\left(0, \frac{1}{16}\right)$, directrix is $y = -\frac{1}{16}$



17. $16x^2 + 25y^2 = 400 \Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1$

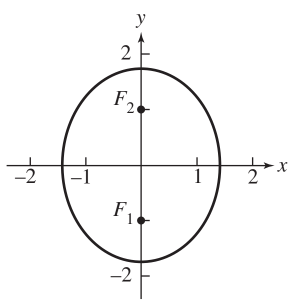
$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{25-16} = 3$

foci are $(\pm 3,\ 0)$



18. $3x^2 + 2y^2 = 6 \Rightarrow \frac{x^2}{2} + \frac{y^2}{3} = 1$

$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{3-2} = 1$ foci are $(0,\ \pm 1)$



19. Foci: $\left(\pm\sqrt{2}, 0\right)$, Vertices:

$(\pm 2,\ 0) \Rightarrow a = 2,\ c = \sqrt{2}$

$\Rightarrow b^2 = a^2 - c^2 = 4 - \left(\sqrt{2}\right)^2 = 2$

$\Rightarrow \frac{x^2}{4} + \frac{y^2}{2} = 1$

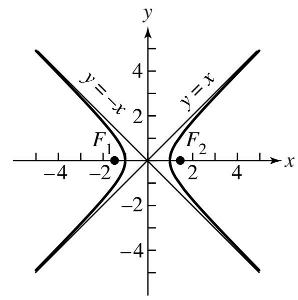
20. Foci: $(0,\ \pm 4)$, Vertices: $(0, \pm 5)$

$\Rightarrow a = 5,\ c = 4 \Rightarrow b^2 = 25 - 16 = 9$

$\Rightarrow \frac{x^2}{9} + \frac{y^2}{25} = 1$

21. $x^2 - y^2 = 1 \Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{1+1} = \sqrt{2}$;

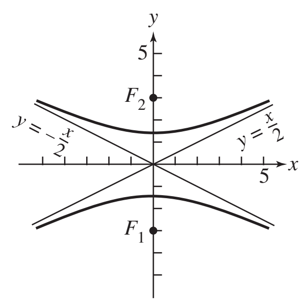
asymptotes are $y = \pm x$, foci are $\left(\pm\sqrt{2}, 0\right)$



22. $8y^2 - 2x^2 = 16 \Rightarrow \frac{y^2}{2} - \frac{x^2}{8} = 1$

$\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{2+8} = \sqrt{10}$;

asymptotes are $y = \pm\frac{x}{2}$, foci are $\left(0, \pm\sqrt{10}\right)$



23. Foci: $\left(0, \pm\sqrt{2}\right)$, Asymptotes: $y = \pm x$

$\Rightarrow c = \sqrt{2}$ and

$\frac{b}{a} = 1 \Rightarrow a = b \Rightarrow c^2 = a^2 + b^2 = 2a^2$

$\Rightarrow 2 = 2a^2 \Rightarrow a = 1 \Rightarrow b = 1$

$\Rightarrow y^2 - x^2 = 1$

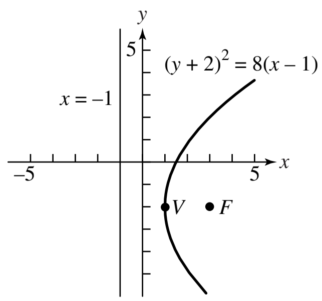
24. Vertices: $(\pm 3,\ 0)$, Asymptotes:

$y = \pm\frac{4}{3}x \Rightarrow a = 3$ and

$\frac{b}{a} = \frac{4}{3} \Rightarrow b = \frac{4}{3}(3) = 4 \Rightarrow \frac{x^2}{9} - \frac{y^2}{16} = 1$

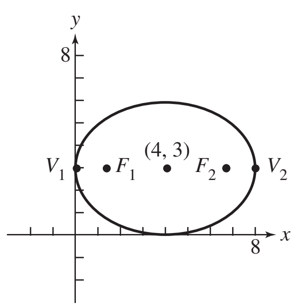
25. **(a)** $y^2 = 8x \Rightarrow 4p = 8 \Rightarrow p = 2 \Rightarrow$ directrix is $x = -2$, focus is (2, 0), and vertex is (0, 0); therefore the new directrix is $x = -1$, the new focus is (3, –2) and the new vertex is (1, –2).

(b)



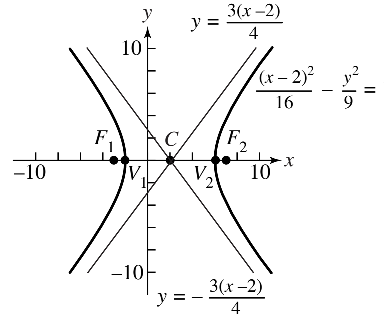
26. **(a)** $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1 \Rightarrow$ center is (0, 0), vertices are (–4, 0) and (4, 0), $c = \sqrt{a^2 - b^2} = \sqrt{7}$ $\Rightarrow$ foci are $\left(\sqrt{7}, 0\right)$ and $\left(-\sqrt{7}, 0\right)$; therefore the new center is (4, 3), the new vertices are (0, 3) and (8, 3), and the new foci are $\left(4 \pm \sqrt{7}, 3\right)$.

(b)



27. **(a)** $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1 \Rightarrow$ center is (0, 0), vertices are (–4, 0) and (4, 0), and the asymptotes are $y = \pm\dfrac{3x}{4}$, $c = \sqrt{a^2 + b^2} = \sqrt{25} = 5$ $\Rightarrow$ foci are (–5, 0) and (5, 0); therefore the new center is (2, 0), the new vertices are (–2, 0) and (6, 0), the new foci are (–3, 0) and (7, 0), and the new asymptotes are $y = \pm\dfrac{3(x-2)}{4}$.

(b)



28. Original parabola: $y^2 = 4x$; vertex is (0, 0); $y^2 = 4x \Rightarrow 4p = 4 \Rightarrow p = 1$, so focus is (1, 0) and directrix is $x = -1$.

New parabola: $(y+3)^2 = 4(x+2)$; vertex is $(-2, -3)$, focus is $(-1, -3)$, directrix is $x = -3$.

29. Original ellipse: $\dfrac{x^2}{6} + \dfrac{y^2}{9} = 1$; vertices are (0, 3) and (0, –3); $c^2 = 9 - 6 \Rightarrow c = \sqrt{3} \Rightarrow$ foci are $(0, \pm\sqrt{3})$; center is (0, 0).

New ellipse: $\dfrac{(x+2)^2}{6} + \dfrac{(y+1)^2}{9} = 1$; vertices are (–2, 2) and (–2, –4); foci are $(-2, -1 \pm \sqrt{3})$; center is $(-2, -1)$.

30. Original hyperbola: $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 1$; vertices are (2, 0) and (–2, 0); $c^2 = 4 + 5 \Rightarrow c = 3 \Rightarrow$ foci are (3, 0) and (–3, 0); center is (0, 0); asymptotes are $y = \pm\dfrac{\sqrt{5}}{2}x$.

New hyperbola: $\dfrac{(x-2)^2}{4} - \dfrac{(y-2)^2}{5} = 1$; vertices are (4, 2) and (0, 2); foci are (5, 2) and (–1, 2); center is (2, 2); asymptotes are $y = \pm\dfrac{\sqrt{5}}{2}(x-2) + 2$.

31. Original hyperbola: $y^2 - x^2 = 1$; vertices are (0, 1) and (0, –1); $c^2 = 1 + 1 \Rightarrow c = \sqrt{2} \Rightarrow$ foci are $\left(0, \pm\sqrt{2}\right)$; center is (0, 0); asymptotes are $y = \pm x$.

New hyperbola: $(y-1)^2 - (x+1)^2 = 1$; vertices are (–1, 2) and (–1, 0); foci are $\left(-1, 1 \pm \sqrt{2}\right)$;

center is (–1, 1); asymptotes are $y = \pm(x+1)+1$.

32. $x^2 + 4x + y^2 = 12 \Rightarrow x^2 + 4x + 4 + y^2 = 12 + 4$ $\Rightarrow (x+2)^2 + y^2 = 16$; this is a circle: center at $C(-2, 0)$, $a = 4$

33. $2x^2 + 2y^2 - 28x + 12y + 114 = 0$ $\Rightarrow x^2 - 14x + 49 + y^2 + 6y + 9 = -57 + 49 + 9$ $\Rightarrow (x-7)^2 + (y+3)^2 = 1$; this is a circle: center at $C(7, -3)$, $a = 1$

34. $x^2 + 2x + 4y - 3 = 0$ $\Rightarrow x^2 + 2x + 1 = -4y + 3 + 1$ $\Rightarrow (x+1)^2 = -4(y-1)$; this is a parabola: $V(-1, 1)$, $F(-1, 0)$

35. $x^2 + 5y^2 + 4x = 1 \Rightarrow x^2 + 4x + 4 + 5y^2 = 4 + 1$ $\Rightarrow (x+2)^2 + 5y^2 = 5 \Rightarrow \frac{(x+2)^2}{5} + y^2 = 1$; this is an ellipse: the center is (–2, 0), the vertices are $\left(-2 \pm \sqrt{5}, 0\right)$; $c = \sqrt{a^2 - b^2} = \sqrt{5-1} = 2$ $\Rightarrow$ the foci are (–4, 0) and (0, 0)

36. $x^2 - y^2 - 2x + 4y = 4$ $\Rightarrow x^2 - 2x + 1 - (y^2 - 4y + 4) = 1$ $\Rightarrow (x-1)^2 - (y-2)^2 = 1$; this is a hyperbola: the center is (1, 2), the vertices are (2, 2) and (0, 2); $c = \sqrt{a^2 + b^2} = \sqrt{1+1} = \sqrt{2} \Rightarrow$ the foci are $(1 \pm \sqrt{2}, 2)$; the asymptotes are $y - 2 = \pm(x-1)$

37. Volume of the Parabolic Solid:

$$V_1 = \int_0^{b/2} 2\pi x\left(h - \frac{4h}{b^2}x^2\right)dx$$
$$= 2\pi h\int_0^{b/2}\left(x - \frac{4x^3}{b^2}\right)dx$$
$$= 2\pi h\left[\frac{x^2}{2} - \frac{x^4}{b^2}\right]_0^{b/2}$$
$$= \frac{\pi hb^2}{8};$$

Volume of the Cone:

$$V_2 = \frac{1}{3}\pi\left(\frac{b}{2}\right)^2 h = \frac{1}{3}\pi\left(\frac{b^2}{4}\right)h = \frac{\pi hb^2}{12};$$

therefore $V_1 = \frac{3}{2}V_2$

38. **(a)** $y^2 = kx \Rightarrow x = \frac{y^2}{k}$; the volume of the solid formed by revolving *A* about the *y*-axis is

$$V_1 = \int_0^{\sqrt{kx}} \pi\left(\frac{y^2}{k}\right)^2 dy$$
$$= \frac{\pi}{k^2}\int_0^{\sqrt{kx}} y^4 dy$$
$$= \frac{\pi x^2\sqrt{kx}}{5};$$

the volume of the right circular cylinder formed by revolving the rectangle about the *y*-axis is $V_2 = \pi x^2\sqrt{kx} \Rightarrow$ the volume of the solid formed by revolving *B* about the *y*-axis is $V_3 = V_2 - V_1 = \frac{4\pi x^2\sqrt{kx}}{5}$.

Therefore we can see the ratio of V_3 to V_1 is 4:1.

(b) The volume of the solid formed by revolving *B* about the *x*-axis is

$$V_1 = \int_0^x \pi\left(\sqrt{kt}\right)^2 dt = \pi k\int_0^x t\ dt = \frac{\pi kx^2}{2}.$$

The volume of the right circular cylinder formed by revolving the rectangle about the *x*-axis is $V_2 = \pi\left(\sqrt{kx}\right)^2 x = \pi kx^2$

$\Rightarrow$ the volume of the solid formed by revolving *A* about the *x*-axis is

$$V_3 = V_2 - V_1 = \pi kx^2 - \frac{\pi kx^2}{2} = \frac{\pi kx^2}{2}.$$

Therefore the ratio of V_3 to V_1 is 1:1.

39. Let $P_1(-p, y_1)$ be any point on $x = -p$, and let $P(x, y)$ be a point where a tangent intersects $y^2 = 4px$. Now $y^2 = 4px \Rightarrow 2y\frac{dy}{dx} = 4p$ $\Rightarrow \frac{dy}{dx} = \frac{2p}{y}$; then the slope of a tangent line from P_1 is $\frac{y - y_1}{x - (-p)} = \frac{dy}{dx} = \frac{2p}{y}$

$\Rightarrow y^2 - yy_1 = 2px + 2p^2$. Since $x = \dfrac{y^2}{4p}$, we have

$$y^2 - yy_1 = 2p\left(\frac{y^2}{4p}\right) + 2p^2$$

$$\Rightarrow y^2 - yy_1 = \frac{1}{2}y^2 + 2p^2$$

$$\Rightarrow \frac{1}{2}y^2 - yy_1 - 2p^2 = 0$$

$$\Rightarrow y^2 - 2y_1(y) - 4p^2 = 0.$$

$$\Rightarrow y = \frac{2y_1 \pm \sqrt{4y_1^2 + 16p^2}}{2} = y_1 \pm \sqrt{y_1^2 + 4p^2}.$$

Therefore the slopes of the two tangents from P_1 are $m_1 = \dfrac{2p}{y_1 + \sqrt{y_1^2 + 4p^2}}$ and

$$m_2 = \frac{2p}{y_1 - \sqrt{y_1^2 + 4p^2}}$$

$\Rightarrow m_1 m_2 = \dfrac{4p^2}{y_1^2 - \left(y_1^2 + 4p^2\right)} = -1 \Rightarrow$ the lines are perpendicular.

40. Let $y = \sqrt{1 - \dfrac{x^2}{4}}$ on the interval $0 \le x \le 2$.

The area of the inscribed rectangle is given by

$A(x) = 2x\left(2\sqrt{1 - \dfrac{x^2}{4}}\right) = 4x\sqrt{1 - \dfrac{x^2}{4}}$ (since the length is $2x$ and the height is $2y$)

$A'(x) = 4\sqrt{1 - \dfrac{x^2}{4}} - \dfrac{x^2}{\sqrt{1 - \frac{x^2}{4}}}$. Thus

$$A'(x) = 0 \Rightarrow 4\sqrt{1 - \frac{x^2}{4}} - \frac{x^2}{\sqrt{1 - \frac{x^2}{4}}} = 0$$

$$\Rightarrow 4\left(1 - \frac{x^2}{4}\right) - x^2 = 0 \Rightarrow x^2 = 2 \Rightarrow x = \sqrt{2}$$

(only the positive square root lies in the interval). Since $A(0) = A(2) = 0$ we have that $A\left(\sqrt{2}\right) = 4$ is the maximum area, when the length is $2\sqrt{2}$ and the height is $\sqrt{2}$.

41. (a) Around the x-axis:

$$9x^2 + 4y^2 = 36 \Rightarrow y^2 = 9 - \frac{9}{4}x^2$$

$\Rightarrow y = \pm\sqrt{9 - \dfrac{9}{4}x^2}$ and we use the positive root

$$\Rightarrow V = 2\int_0^2 \pi\left(\sqrt{9 - \frac{9}{4}x^2}\right)^2 dx = 2\int_0^2 \pi\left(9 - \frac{9}{4}x^2\right)dx = 2\pi\left[9x - \frac{3}{4}x^3\right]_0^2 = 24\pi$$

(b) Around the y-axis:

$$9x^2 + 4y^2 = 36 \Rightarrow x^2 = 4 - \frac{4}{9}y^2$$

$\Rightarrow x = \pm\sqrt{4 - \dfrac{4}{9}y^2}$ and we use the positive root

$$\Rightarrow V = 2\int_0^3 \pi\left(\sqrt{4 - \frac{4}{9}y^2}\right)^2 dy = 2\int_0^3 \pi\left(4 - \frac{4}{9}y^2\right) dy = 2\pi\left[4y - \frac{4}{27}y^3\right]_0^3 = 16\pi$$

42. $9x^2 - 4y^2 = 36 \Rightarrow y^2 = \dfrac{9x^2 - 36}{4}$ on the

$\Rightarrow y = \pm\dfrac{3}{2}\sqrt{x^2 - 4}$

interval

$$2 \le x \le 4 \Rightarrow V = \int_2^4 \pi\left(\frac{3}{2}\sqrt{x^2-4}\right)^2 dx$$
$$= \frac{9\pi}{4}\int_2^4 (x^2-4)\,dx$$
$$= \frac{9\pi}{4}\left[\frac{x^3}{3}-4x\right]_2^4$$
$$= \frac{9\pi}{4}\left[\left(\frac{64}{3}-16\right)-\left(\frac{8}{3}-8\right)\right]$$
$$= \frac{9\pi}{4}\left(\frac{56}{3}-8\right)$$
$$= \frac{3\pi}{4}(56-24)$$
$$= 24\pi$$

43. $x^2 - y^2 = 1 \Rightarrow x = \pm\sqrt{1+y^2}$ on the interval $-3 \le y \le 3$

$$\Rightarrow V = \int_{-3}^{3} \pi\left(\sqrt{1+y^2}\right)^2 dy$$
$$= 2\int_0^3 \pi\left(\sqrt{1+y^2}\right)^2 dy$$
$$= 2\pi\int_0^3 (1+y^2)dy$$
$$= 2\pi\left[y+\frac{y^3}{3}\right]_0^3$$
$$= 24\pi$$

44. $y = \int \frac{w}{H}x\,dx = \frac{w}{H}\left(\frac{x^2}{2}\right)+C = \frac{wx^2}{2H}+C;$

$y = 0$ when $x = 0 \Rightarrow 0 = \frac{w(0)^2}{2H}+C \Rightarrow C = 0;$

therefore $y = \frac{wx^2}{2H}$ is the equation of the cable's curve.

45. $\frac{dr_A}{dt} = \frac{dr_B}{dt} \Rightarrow \frac{d}{dt}(r_A - r_B) = 0$

$\Rightarrow r_A - r_B =$ a constant

46. PF will always equal PB because the string has constant length $AB = FP + PA = AP + PB$.

Appendix A5.2 (pp. 598–603)

Exercises

1. $16x^2 + 25y^2 = 400 \Rightarrow \frac{x^2}{25}+\frac{y^2}{16} = 1$

$\Rightarrow c = \sqrt{a^2-b^2} = \sqrt{25-16} = 3$

$\Rightarrow e = \frac{c}{a} = \frac{3}{5};$ $F(\pm 3,\ 0)$; directrices are

$x = 0 \pm \frac{a}{e} = \pm\frac{5}{\left(\frac{3}{5}\right)} = \pm\frac{25}{3}$

2. $2x^2 + y^2 = 2 \Rightarrow x^2 + \frac{y^2}{2} = 1$

$\Rightarrow c = \sqrt{a^2-b^2} = \sqrt{2-1} = 1$

$\Rightarrow e = \frac{c}{a} = \frac{1}{\sqrt{2}};$ $F(0, \pm 1)$; directrices are

$y = 0 \pm \frac{a}{e} = \pm\frac{\sqrt{2}}{\left(\frac{1}{\sqrt{2}}\right)} = \pm 2$

3. $3x^2 + 2y^2 = 6 \Rightarrow \frac{x^2}{2}+\frac{y^2}{3} = 1$

$\Rightarrow c = \sqrt{a^2-b^2} = \sqrt{3-2} = 1 \Rightarrow e = \frac{c}{a} = \frac{1}{\sqrt{3}};$

$F(0, \pm 1)$; directrices are

$y = 0 \pm \frac{a}{e} = \pm\frac{\sqrt{3}}{\left(\frac{1}{\sqrt{3}}\right)} = \pm 3$

4. $6x^2 + 9y^2 = 54 \Rightarrow \frac{x^2}{9}+\frac{y^2}{6} = 1$

$\Rightarrow c = \sqrt{a^2-b^2} = \sqrt{9-6} = \sqrt{3}$

$\Rightarrow e = \frac{c}{a} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}};$ $F(\pm\sqrt{3},\ 0)$; directrices are $x = 0 \pm \frac{a}{e} = \pm\frac{3}{\left(\frac{1}{\sqrt{3}}\right)} = \pm 3\sqrt{3}$

5. Foci: $(0, \pm 3)$, $e = 0.5 \Rightarrow c = 3$ and

$a = \frac{c}{e} = \frac{3}{0.5} = 6 \Rightarrow b^2 = 36 - 9 = 27$

$\Rightarrow \frac{x^2}{27}+\frac{y^2}{36} = 1$

6. Foci: $(\pm 8, 0)$, $e = 0.2 \Rightarrow c = 8$ and

$$a = \frac{c}{e} = \frac{8}{0.2} = 40 \Rightarrow b^2 = 1600 - 64 = 1536$$
$$\Rightarrow \frac{x^2}{1600} + \frac{y^2}{1536} = 1$$

7. Vertices : $(\pm 10, 0)$, $e = 0.24 \Rightarrow a = 10$ and

$$c = ae = 10(0.24) = 2.4$$
$$\Rightarrow b^2 = 100 - 5.76 = 94.24$$
$$\Rightarrow \frac{x^2}{100} + \frac{y^2}{94.24} = 1$$

8. Vertices: $(0, \pm 70)$, $e = 0.1 \Rightarrow a = 70$ and

$$c = ae = 70(0.1) = 7 \Rightarrow b^2 = 4900 - 49 = 4851$$
$$\Rightarrow \frac{x^2}{4851} + \frac{y^2}{4900} = 1$$

9. Focus: $\left(\sqrt{5}, 0\right)$, Directrix:

$$x = \frac{9}{\sqrt{5}} \Rightarrow c = ae = \sqrt{5} \text{ and}$$
$$\frac{a}{e} = \frac{9}{\sqrt{5}} \Rightarrow \frac{ae}{e^2} = \frac{9}{\sqrt{5}} \Rightarrow \frac{\sqrt{5}}{e^2} = \frac{9}{\sqrt{5}} \Rightarrow e^2 = \frac{5}{9}$$
$$\Rightarrow e = \frac{\sqrt{5}}{3}.$$

Then

$$PF = \frac{\sqrt{5}}{3} PD$$
$$\Rightarrow \sqrt{\left(x - \sqrt{5}\right)^2 + (y - 0)^2} = \frac{\sqrt{5}}{3} \left| x - \frac{9}{\sqrt{5}} \right|$$
$$\Rightarrow \left(x - \sqrt{5}\right)^2 + y^2 = \frac{5}{9}\left(x - \frac{9}{\sqrt{5}} \right)^2$$
$$\Rightarrow x^2 - 2\sqrt{5}x + 5 + y^2 = \frac{5}{9}\left(x^2 - \frac{18}{\sqrt{5}} x + \frac{81}{5} \right)$$
$$\Rightarrow \frac{4}{9}x^2 + y^2 = 4 \Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1$$

10. Focus: $(-4, 0)$, Directrix:

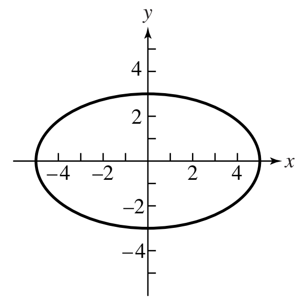
$$x = -16 \Rightarrow c = ae = 4 \text{ and}$$
$$\frac{a}{e} = 16 \Rightarrow \frac{ae}{e^2} = 16 \Rightarrow \frac{4}{e^2} = 16 \Rightarrow e^2 = \frac{1}{4}$$
$$\Rightarrow e = \frac{1}{2}.$$

Then

$$PF = \frac{1}{2} PD \Rightarrow \sqrt{(x+4)^2 + (y-0)^2} = \frac{1}{2}\left| x + 16 \right|$$
$$\Rightarrow (x+4)^2 + y^2 = \frac{1}{4}(x+16)^2$$
$$\Rightarrow x^2 + 8x + 16 + y^2 = \frac{1}{4}(x^2 + 32x + 256)$$
$$\Rightarrow \frac{3}{4}x^2 + y^2 = 48 \Rightarrow \frac{x^2}{64} + \frac{y^2}{48} = 1$$

11. $e = \frac{4}{5} \Rightarrow$ take $c = 4$ and $a = 5$;

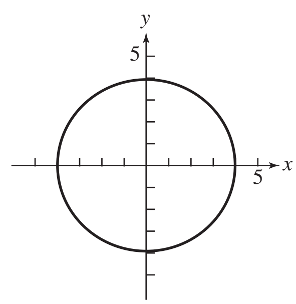
$$c^2 = a^2 - b^2 \Rightarrow 16 = 25 - b^2 \Rightarrow b^2 = 9$$
$$\Rightarrow b = 3; \text{ therefore } \frac{x^2}{25} + \frac{y^2}{9} = 1$$



12. The eccentricity e for Pluto is 0.25,

$$\Rightarrow e = \frac{c}{a} = 0.25 = \frac{1}{4} \text{ take } c = 1 \text{ and } a = 4;$$
$$c^2 = a^2 - b^2 \Rightarrow 1 = 16 - b^2$$
$$\Rightarrow b^2 = 15 \Rightarrow b = \sqrt{15}; \text{ therefore,}$$

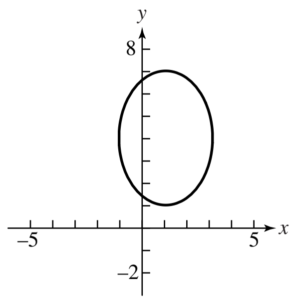
$\frac{x^2}{16} + \frac{y^2}{15} = 1$ is a model of Pluto's orbit.



13. One axis is from (1, 1) to (1, 7) and is 6 units long; the other axis is from (3, 4) to (–1, 4) and is 4 units long. Therefore, $a = 3$, $b = 2$ and the major axis is vertical. The center is the point $C(1, 4)$ and the ellipse is given by

$$\frac{(x-1)^2}{4} + \frac{(y-4)^2}{9} = 1;$$
$$c^2 = a^2 - b^2 = 3^2 - 2^2 = 5 \Rightarrow c = \sqrt{5};$$

therefore the foci are $F\left(1, 4 \pm \sqrt{5}\right)$, the eccentricity is $e = \frac{c}{a} = \frac{\sqrt{5}}{3}$, and the directrices are $y = 4 \pm \frac{a}{e} = 4 \pm \frac{3}{\left(\frac{\sqrt{5}}{3}\right)} = 4 \pm \frac{9\sqrt{5}}{5}$.



14. Using $PF = e \cdot PD$, we have

$$\sqrt{(x-4)^2 + y^2} = \frac{2}{3}|x-9|$$
$$\Rightarrow (x-4)^2 + y^2 = \frac{4}{9}(x-9)^2$$
$$\Rightarrow x^2 - 8x + 16 + y^2 = \frac{4}{9}(x^2 - 18x + 81)$$
$$\Rightarrow \frac{5}{9}x^2 + y^2 = 20 \Rightarrow 5x^2 + 9y^2 = 180$$

or $\frac{x^2}{36} + \frac{y^2}{20} = 1$.

15. $9x^2 - 16y^2 = 144 \Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1$

$\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{16+9} = 5 \Rightarrow e = \frac{c}{a} = \frac{5}{4}$;

asymptotes are $y = \pm\frac{3}{4}x$; $F(\pm 5, 0)$; directrices are $x = 0 \pm \frac{a}{e} = \pm\frac{16}{5}$.

16. $y^2 - x^2 = 8 \Rightarrow \frac{y^2}{8} - \frac{x^2}{8} = 1$

$\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{8+8} = 4$

$\Rightarrow e = \frac{c}{a} = \frac{4}{\sqrt{8}} = \sqrt{2}$;

asymptotes are $y = \pm x$; $F(0, \pm 4)$; directrices are $y = 0 \pm \frac{a}{e} = \pm\frac{\sqrt{8}}{\sqrt{2}} = \pm 2$

17. $8x^2 - 2y^2 = 16 \Rightarrow \frac{x^2}{2} - \frac{y^2}{8} = 1$

$\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{2+8} = \sqrt{10}$

$\Rightarrow e = \frac{c}{a} = \frac{\sqrt{10}}{\sqrt{2}} = \sqrt{5}$;

asymptotes are $y = \pm 2x$; $F\left(\pm\sqrt{10}, 0\right)$;

directrices are $x = 0 \pm \frac{a}{e} = \pm\frac{\sqrt{2}}{\sqrt{5}} = \pm\frac{\sqrt{10}}{5}$

18. $8y^2 - 2x^2 = 16 \Rightarrow \frac{y^2}{2} - \frac{x^2}{8} = 1$

$\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{2+8} = \sqrt{10}$

$\Rightarrow e = \frac{c}{a} = \frac{\sqrt{10}}{\sqrt{2}} = \sqrt{5}$;

asymptotes are $y = \pm\frac{x}{2}$; $F\left(0, \pm\sqrt{10}\right)$;

directrices are $y = 0 \pm \frac{a}{e} = \pm\frac{\sqrt{2}}{\sqrt{5}} = \pm\frac{2}{\sqrt{10}}$

19. Vertices: $(0, \pm 1)$ and $e = 3 \Rightarrow a = 1$ and

$e = \frac{c}{a} = 3 \Rightarrow c = 3a = 3$

$\Rightarrow b^2 = c^2 - a^2 = 9 - 1 = 8$

$\Rightarrow y^2 - \frac{x^2}{8} = 1$

20. Foci $(\pm 3, 0)$ and $e = 3 \Rightarrow c = 3$ and

$e = \frac{c}{a} = 3 \Rightarrow c = 3a \Rightarrow a = 1$

$\Rightarrow b^2 = c^2 - a^2 = 9 - 1 = 8 \Rightarrow x^2 - \frac{y^2}{8} = 1$

21. Focus: (4, 0), directrix: $x = 2 \Rightarrow c = ae = 4$ and $\frac{a}{e} = 2 \Rightarrow \frac{ae}{e^2} = 2 \Rightarrow \frac{4}{e^2} = 2$

$\Rightarrow e^2 = 2 \Rightarrow e = \sqrt{2}$.

Then

$PF = \sqrt{2}PD$

$\Rightarrow \sqrt{(x-4)^2 + (y-0)^2} = \sqrt{2}|x-2|$

$\Rightarrow (x-4)^2 + y^2 = 2(x-2)^2$

$\Rightarrow x^2 - 8x + 16 + y^2 = 2(x^2 - 4x + 4)$

$\Rightarrow -x^2 + y^2 = -8 \Rightarrow \frac{x^2}{8} - \frac{y^2}{8} = 1$

22. Focus: (–2, 0), directrix: $x=-\frac{1}{2}\Rightarrow c=ae=2$
and $\frac{a}{e}=\frac{1}{2}\Rightarrow\frac{ae}{e^2}=\frac{1}{2}\Rightarrow\frac{2}{e^2}=\frac{1}{2}\Rightarrow e^2=4$
$\Rightarrow e=2.$
Then
$$PF=2PD\Rightarrow\sqrt{(x+2)^2+(y-0)^2}=2\left|x+\frac{1}{2}\right|$$
$$\Rightarrow(x+2)^2+y^2=4\left(x+\frac{1}{2}\right)^2$$
$$\Rightarrow x^2+4x+4+y^2=4\left(x^2+x+\frac{1}{4}\right)$$
$$\Rightarrow -3x^2+y^2=-3\Rightarrow x^2-\frac{y^2}{3}=1$$

23. $\sqrt{(x-1)^2+(y+3)^2}=\frac{3}{2}|y-2|$
$$\Rightarrow x^2-2x+1+y^2+6y+9=\frac{9}{4}(y^2-4y+4)$$
$$\Rightarrow 4x^2-5y^2-8x+60y+4=0$$
$$\Rightarrow 4(x^2-2x+1)-5(y^2-12y+36)$$
$$=-4+4-180$$
$$\Rightarrow\frac{(y-6)^2}{36}-\frac{(x-1)^2}{45}=1$$

24. $c^2=a^2+b^2\Rightarrow b^2=c^2-a^2;\ e=\frac{c}{a}\Rightarrow c=ea$
$\Rightarrow c^2=e^2a^2\Rightarrow b^2=e^2a^2-a^2=a^2(e^2-1);$
thus, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\Rightarrow\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1;$ the asymptotes of this hyperbola are $y=\pm\sqrt{e^2-1}\cdot x.$ As e increases, the slopes of the asymptotes increase and the hyperbolas approach a single vertical line (the y-axis).

25. The ellipse must pass through $(0,0)\Rightarrow c=0;$ the point (–1, 2) lies on the ellipse $\Rightarrow -a+2b=-8.$ The ellipse is tangent to the x-axis $\Rightarrow$ its center is on the y-axis, so $a=0$ and $b=-4\Rightarrow$ the equation is $4x^2+y^2-4y=0.$ Next,
$$4x^2+y^2-4y+4=4\Rightarrow 4x^2+(y-2)^2=4$$
$\Rightarrow x^2+\frac{(y-2)^2}{4}=1\Rightarrow a=2$ and $b=1$ (now using the standard symbols)
$$\Rightarrow c^2=a^2-b^2=4-1=3$$
$$\Rightarrow c=\sqrt{3}\Rightarrow e=\frac{c}{a}=\frac{\sqrt{3}}{2}.$$

26. We first prove a result which we will use: let m_1 and m_2 be the slopes of two nonparallel, nonperpendicular lines. m_1 is the slope of the line with the larger angle of inclination. Let be the acute angle between the lines. Then $\tan\alpha=\frac{m_1-m_2}{1+m_1m_2}$. (To see this result for positive-slope lines, let θ_1 be the angle of inclination of the line with slope m_1, and θ_2 be the angle of inclination of the line with slope m_2. Assume $m_1>m_2$. Then $\theta_1>\theta_2$ and we have $\alpha=\theta_1-\theta_2$. Then
$$\tan\alpha=\tan(\theta_1-\theta_2)$$
$$=\frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_1\tan\theta_2}$$
$$=\frac{m_1-m_2}{1+m_1m_2},$$
since $m_1=\tan\theta_1$ and $m_2=\tan\theta_2$.)
We can write the equation of the ellipse as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with $a>b$, and then $b^2x^2+a^2y^2=a^2b^2.$
Now we prove the reflective property of ellipses (see the accompanying figure):
$$2b^2x+2a^2yy'=0\Rightarrow y'=-\frac{b^2x}{a^2y}.$$
Let $P(x_0,\,y_0)$ be any point on the ellipse $y'(x_0)=-\frac{b^2x_0}{a^2y_0}$. Let $F_1(c,0)$ and $F_2(c,0)$ be the foci.
Then $m_{PF_1}=\frac{y_0}{x_0-c}$ and $m_{PF_2}=\frac{y_0}{x_0+c}$. Let α and β be the angles between the tangent line and PF_1 and PF_2 respectively. Then

$$\tan\alpha = \frac{\left(-\frac{b^2x_0}{a^2y_0} - \frac{y_0}{x_0-c}\right)}{\left(1-\frac{b^2x_0y_0}{a^2y_0(x_0-c)}\right)}$$

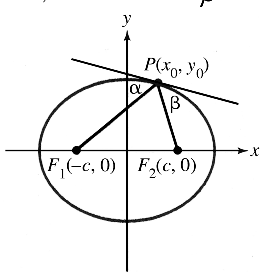
$$= \frac{-b^2x_0{}^2 + b^2x_0c - a^2y_0{}^2}{a^2y_0x_0 - a^2y_0c - b^2x_0y_0}$$

$$= \frac{b^2x_0c - \left(b^2x_0{}^2 + a^2y_0{}^2\right)}{-a^2y_0c + (a^2-b^2)x_0y_0}$$

$$= \frac{b^2x_0c - a^2b^2}{-a^2y_0c + c^2x_0y_0}$$

$$= \frac{b^2}{cy_0}.$$

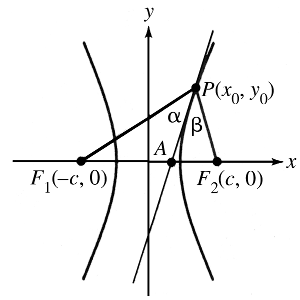
Similarly, $\tan\beta = \frac{b^2}{cy_0}$. Since $\tan\alpha = \tan\beta$, and α and β are both less than 90°, we have $\alpha = \beta$.



27. To prove the reflective property for hyperbolas:

$$b^2x^2 - a^2y^2 = a^2b^2$$
$$2b^2x - 2a^2yy' = 0$$
$$y' = \frac{b^2x}{a^2y}$$

Let $P(x_0, y_0)$ be a point of tangency (see the accompanying figure). The slope from P to $F_1(-c, 0)$ is $\frac{y_0}{x_0+c}$ and from P to $F_2(c, 0)$ it is $\frac{y_0}{x_0-c}$. Let the tangent through P meet the x-axis in point A, and define the angles $\angle F_1PA = \alpha$ and $\angle F_2PA = \beta$. We will show that $\tan\alpha = \tan\beta$.



From the preliminary result in Exercise 26,

$$\tan\alpha = \frac{\left(\frac{x_0b^2}{y_0a^2} - \frac{y_0}{x_0+c}\right)}{1+\left(\frac{x_0b^2}{y_0a^2}\right)\left(\frac{y_0}{x_0+c}\right)}$$

$$= \frac{x_0{}^2b^2 + x_0b^2c - y_0{}^2a^2}{x_0y_0a^2 + y_0a^2c + x_0y_0b^2}$$

$$= \frac{a^2b^2 + x_0b^2c}{x_0y_0c^2 + y_0a^2c}$$

$$= \frac{b^2}{y_0c}.$$

In a similar manner,

$$\tan\beta = \frac{\left(\frac{y_0}{x_0-c} - \frac{x_0b^2}{y_0a^2}\right)}{1+\left(\frac{y_0}{x_0-c}\right)\left(\frac{x_0b^2}{y_0a^2}\right)} = \frac{b^2}{y_0c}.$$

Since $\tan\alpha = \tan\beta$, and α and β are acute angles, we have $\alpha = \beta$.

28. The tangent to the ellipse at P bisects $\angle APC$, and the tangent to the hyperbola at P bisects $\angle APB$. Since $\angle APC$ and $\angle APB$ are a linear pair, so that $m\angle APC + m\angle APB = 180°$ and $\frac{m\angle APC}{2} + \frac{m\angle APB}{2} = 90°$, the tangents to the ellipse and hyperbola are perpendicular.

Appendix A5.3 (pp. 604–610)

Exercises

1. $x^2 - 3xy + y^2 - x = 0$

$\Rightarrow B^2 - 4AC = (-3)^2 - 4(1)(1) = 5 > 0$
$\Rightarrow$ Hyperbola

2. $3x^2 - 18xy + 27y^2 - 5x + 7y = -4$

$\Rightarrow B^2 - 4AC = (-18)^2 - 4(3)(27) = 0$
$\Rightarrow$ Parabola

3. $3x^2 - 7xy + \sqrt{17}y^2 = 1$
$\Rightarrow B^2 - 4AC = (-7)^2 - 4(3)\sqrt{17} \approx -0.477 < 0$
$\Rightarrow$ Ellipse

4. $2x^2 - \sqrt{15}xy + 2y^2 + x + y = 0$
$\Rightarrow B^2 - 4AC = \left(-\sqrt{15}\right)^2 - 4(2)(2) = -1 < 0$
$\Rightarrow$ Ellipse

5. $x^2 + 2xy + y^2 + 2x - y + 2 = 0$
$\Rightarrow B^2 - 4AC = 2^2 - 4(1)(1) = 0 \Rightarrow$ Parabola

6. $2x^2 - y^2 + 4xy - 2x + 3y = 6$
$\Rightarrow B^2 - 4AC = 4^2 - 4(2)(-1) = 24 > 0$
$\Rightarrow$ Hyperbola

7. $x^2 + 4xy + 4y^2 - 3x = 6$
$\Rightarrow B^2 - 4AC = 4^2 - 4(1)(4) = 0 \Rightarrow$ Parabola

8. $x^2 + y^2 + 3x - 2y = 10$
$\Rightarrow B^2 - 4AC = 0^2 - 4(1)(1) = -4 < 0$
$\Rightarrow$ Ellipse (circle)

9. $xy + y^2 - 3x = 5$
$\Rightarrow B^2 - 4AC = 1^2 - 4(0)(1) = 1 > 0$
$\Rightarrow$ Hyperbola

10. $3x^2 + 6xy + 3y^2 - 4x + 5y = 12$
$\Rightarrow B^2 - 4AC = 6^2 - 4(3)(3) = 0 \Rightarrow$ Parabola

11. $3x^2 - 5xy + 2y^2 - 7x - 14y = -1$
$\Rightarrow B^2 - 4AC = (-5)^2 - 4(3)(2) = 1 > 0$
$\Rightarrow$ Hyperbola

12. $2x^2 - 4.9xy + 3y^2 - 4x = 7$
$\Rightarrow B^2 - 4AC = (-4.9)^2 - 4(2)(3) = 0.01 > 0$
$\Rightarrow$ Hyperbola

13. $x^2 - 3xy + 3y^2 + 6y = 7$
$\Rightarrow B^2 - 4AC = (-3)^2 - 4(1)(3) = -3 < 0$
$\Rightarrow$ Ellipse

14. $25x^2 + 21xy + 4y^2 - 350x = 0$
$\Rightarrow B^2 - 4AC = 21^2 - 4(25)(4) = 41 > 0$
$\Rightarrow$ Hyperbola

15. $6x^2 + 3xy + 2y^2 + 17y + 2 = 0$
$\Rightarrow B^2 - 4AC = 3^2 - 4(6)(2) = -39 < 0$
$\Rightarrow$ Ellipse

16. $3x^2 + 12xy + 12y^2 + 435x - 9y + 72 = 0$
$\Rightarrow B^2 - 4AC = 12^2 - 4(3)(12) = 0 \Rightarrow$ Parabola

17. $\cot 2\alpha = \frac{A-C}{B} = \frac{0}{1} = 0 \Rightarrow 2\alpha = \frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{4}$;
therefore $x = x' \cos\alpha - y' \sin\alpha$,
$y = x' \sin\alpha + y' \cos\alpha$
$\Rightarrow x = \frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y',\ y = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$
$\Rightarrow \left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right)\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right) = 2$
$\Rightarrow \frac{1}{2}x'^2 - \frac{1}{2}y'^2 = 2 \Rightarrow x'^2 - y'^2 = 4$
$\Rightarrow$ Hyperbola

18. $\cot 2\alpha = \frac{A-C}{B} = \frac{1-1}{1} = 0$

$\Rightarrow 2\alpha = \frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{4},$

therefore,

$x = x'\cos\alpha - y'\sin\alpha,\ y = x'\sin\alpha + y'\cos\alpha$

$\Rightarrow x = \frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y',\ y = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$

$$\Rightarrow \left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right)^2 + \left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right)\cdot\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right) + \left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right)^2 = 1$$

$$\Rightarrow \frac{1}{2}x'^2 - x'y' + \frac{1}{2}y'^2 + \frac{1}{2}x'^2 - \frac{1}{2}y'^2 + \frac{1}{2}x'^2 + x'y' + \frac{1}{2}y'^2 = 1$$

$$\Rightarrow \frac{3}{2}x'^2 + \frac{1}{2}y'^2 = 1 \Rightarrow 3x'^2 + y'^2 = 2$$

$\Rightarrow$ Ellipse

19. $\cot 2\alpha = \frac{A-C}{B} = \frac{3-1}{2\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow 2\alpha = \frac{\pi}{3} \Rightarrow \alpha = \frac{\pi}{6};$ therefore

$x = x'\cos\alpha - y'\sin\alpha,\ y = x'\sin\alpha + y'\cos\alpha$

$\Rightarrow x = \frac{\sqrt{3}}{2}x' - \frac{1}{2}y',\ y = \frac{1}{2}x' + \frac{\sqrt{3}}{2}y'$

$$\Rightarrow 3\left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y'\right)^2 + 2\sqrt{3}\left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y'\right)\left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right) + \left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right)^2 - 8\left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y'\right)$$
$$+ 8\sqrt{3}\left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right) = 0$$

$\Rightarrow 4x'^2 + 16y' = 0$

$\Rightarrow$ Parabola

20. $\cot 2\alpha = \frac{A-C}{B} = \frac{1-2}{-\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow 2\alpha = \frac{\pi}{3} \Rightarrow \alpha = \frac{\pi}{6};$ therefore

$x = x'\cos\alpha - y'\sin\alpha,\ y = x'\sin\alpha + y'\cos\alpha$

$\Rightarrow x = \frac{\sqrt{3}}{2}x' - \frac{1}{2}y',\ y = \frac{1}{2}x' + \frac{\sqrt{3}}{2}y'$

$$\Rightarrow \left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y'\right)^2 - \sqrt{3}\left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y'\right)\left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right) + 2\left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right)^2 = 1$$

$$\Rightarrow \frac{1}{2}x'^2 + \frac{5}{2}y'^2 = 1$$

$\Rightarrow x'^2 + 5y'^2 = 2$

$\Rightarrow$ Ellipse

21. $\cot 2\alpha = \dfrac{A-C}{B} = \dfrac{1-1}{-2} = 0 \Rightarrow 2\alpha = \dfrac{\pi}{2} \Rightarrow \alpha = \dfrac{\pi}{4}$; therefore

$x = x' \cos\alpha - y' \sin\alpha,\ y = x' \sin\alpha + y' \cos\alpha$

$\Rightarrow x = \dfrac{\sqrt{2}}{2}x' - \dfrac{\sqrt{2}}{2}y',\ y = \dfrac{\sqrt{2}}{2}x' + \dfrac{\sqrt{2}}{2}y'$

$\Rightarrow \left(\dfrac{\sqrt{2}}{2}x' - \dfrac{\sqrt{2}}{2}y'\right)^2 - 2\left(\dfrac{\sqrt{2}}{2}x' - \dfrac{\sqrt{2}}{2}y'\right)\left(\dfrac{\sqrt{2}}{2}x' + \dfrac{\sqrt{2}}{2}y'\right) + \left(\dfrac{\sqrt{2}}{2}x' + \dfrac{\sqrt{2}}{2}y'\right)^2 = 2$

$\Rightarrow 2y'^2 = 2$

$\Rightarrow y'^2 = 1$

$\Rightarrow$ Parallel horizontal lines

22. $\cot 2\alpha = \dfrac{A-C}{B} = \dfrac{3-1}{-2\sqrt{3}} = -\dfrac{1}{\sqrt{3}} \Rightarrow 2\alpha = \dfrac{2\pi}{3} \Rightarrow \alpha = \dfrac{\pi}{3}$; therefore

$x = x' \cos\alpha - y' \sin\alpha,\ y = x' \sin\alpha + y' \cos\alpha$

$\Rightarrow x = \dfrac{1}{2}x' - \dfrac{\sqrt{3}}{2}y',\ y = \dfrac{\sqrt{3}}{2}x' + \dfrac{1}{2}y'$

$\Rightarrow 3\left(\dfrac{1}{2}x' - \dfrac{\sqrt{3}}{2}y'\right)^2 - 2\sqrt{3}\left(\dfrac{1}{2}x' - \dfrac{\sqrt{3}}{2}y'\right)\left(\dfrac{\sqrt{3}}{2}x' + \dfrac{1}{2}y'\right) + \left(\dfrac{\sqrt{3}}{2}x' + \dfrac{1}{2}y'\right)^2 = 1$

$\Rightarrow 4y'^2 = 1$

$\Rightarrow$ Parallel horizontal lines

23. $\cot 2\alpha = \dfrac{A-C}{B} = \dfrac{\sqrt{2}-\sqrt{2}}{2\sqrt{2}} = 0 \Rightarrow 2\alpha = \dfrac{\pi}{2} \Rightarrow \alpha = \dfrac{\pi}{4}$; therefore

$x = x' \cos\alpha - y' \sin\alpha,\ y = x' \sin\alpha + y' \cos\alpha$

$\Rightarrow x = \dfrac{\sqrt{2}}{2}x' - \dfrac{\sqrt{2}}{2}y',\ y = \dfrac{\sqrt{2}}{2}x' + \dfrac{\sqrt{2}}{2}y'$

$\Rightarrow \sqrt{2}\left(\dfrac{\sqrt{2}}{2}x' - \dfrac{\sqrt{2}}{2}y'\right)^2 + 2\sqrt{2}\left(\dfrac{\sqrt{2}}{2}x' - \dfrac{\sqrt{2}}{2}y'\right)\left(\dfrac{\sqrt{2}}{2}x' + \dfrac{\sqrt{2}}{2}y'\right) + \sqrt{2}\left(\dfrac{\sqrt{2}}{2}x' + \dfrac{\sqrt{2}}{2}y'\right)^2 - 8\left(\dfrac{\sqrt{2}}{2}x' - \dfrac{\sqrt{2}}{2}y'\right)$

$\quad + 8\left(\dfrac{\sqrt{2}}{2}x' + \dfrac{\sqrt{2}}{2}y'\right) = 0$

$\Rightarrow 2\sqrt{2}x'^2 + 8\sqrt{2}y' = 0$

$\Rightarrow x'^2 + 4y' = 0$

$\Rightarrow$ Parabola

24. $\cot 2\alpha = \dfrac{A-C}{B} = \dfrac{0-0}{1} = 0 \Rightarrow 2\alpha = \dfrac{\pi}{2} \Rightarrow \alpha = \dfrac{\pi}{4}$; therefore

$x = x' \cos\alpha - y' \sin\alpha,\ y = x' \sin\alpha + y' \cos\alpha$

$\Rightarrow x = \dfrac{\sqrt{2}}{2}x' - \dfrac{\sqrt{2}}{2}y',\ y = \dfrac{\sqrt{2}}{2}x' + \dfrac{\sqrt{2}}{2}y'$

$\Rightarrow \left(\dfrac{\sqrt{2}}{2}x' - \dfrac{\sqrt{2}}{2}y'\right)\left(\dfrac{\sqrt{2}}{2}x' + \dfrac{\sqrt{2}}{2}y'\right) - \left(\dfrac{\sqrt{2}}{2}x' + \dfrac{\sqrt{2}}{2}y'\right) - \left(\dfrac{\sqrt{2}}{2}x' - \dfrac{\sqrt{2}}{2}y'\right) + 1 = 0$

$\Rightarrow \dfrac{1}{2}x'^2 - \dfrac{1}{2}y'^2 - \sqrt{2}x' + 1 = 0$

$\Rightarrow x'^2 - y'^2 - 2\sqrt{2}x' + 2 = 0$

$\Rightarrow$ Hyperbola

25. $\cot 2\alpha = \frac{A-C}{B} = \frac{3-3}{2} = 0 \Rightarrow 2\alpha = \frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{4}$; therefore
$x = x'\cos\alpha - y'\sin\alpha,\ y = x'\sin\alpha + y'\cos\alpha$
$\Rightarrow x = \frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y',\ y = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$
$\Rightarrow 3\left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right)^2 + 2\left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right)\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right) + 3\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right)^2 = 19$
$\Rightarrow 4x'^2 + 2y'^2 = 19$
$\Rightarrow$ Ellipse

26. $\cot 2\alpha = \frac{A-C}{B} = \frac{3-(-1)}{4\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow 2\alpha = \frac{\pi}{3} \Rightarrow \alpha = \frac{\pi}{6}$; therefore
$x = x'\cos\alpha - y'\sin\alpha,\ y = x'\sin\alpha + y'\cos\alpha$
$\Rightarrow x = \frac{\sqrt{3}}{2}x' - \frac{1}{2}y',\ y = \frac{1}{2}x' + \frac{\sqrt{3}}{2}y'$
$\Rightarrow 3\left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y'\right)^2 + 4\sqrt{3}\left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y'\right)\left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right) - \left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right)^2 = 7$
$\Rightarrow 5x'^2 - 3y'^2 = 7$
$\Rightarrow$ Hyperbola

27. $\cot 2\alpha = \frac{A-C}{B} = \frac{14-2}{16} = \frac{3}{4} \Rightarrow \cos 2\alpha = \frac{3}{5}$ (if we choose 2α in Quadrant I);
thus $\sin\alpha = \sqrt{\frac{1-\cos 2\alpha}{2}} = \sqrt{\frac{1-\left(\frac{3}{5}\right)}{2}} = \frac{1}{\sqrt{5}}$ and $\cos\alpha = \sqrt{\frac{1+\cos 2\alpha}{2}} = \sqrt{\frac{1+\left(\frac{3}{5}\right)}{2}} = \frac{2}{\sqrt{5}}$
$\left(\text{or } \sin\alpha = -\frac{2}{\sqrt{5}} \text{ and } \cos\alpha = \frac{1}{\sqrt{5}}\right)$

28. $\cot 2\alpha = \frac{A-C}{B} = \frac{4-1}{-4} = -\frac{3}{4} \Rightarrow \cos 2\alpha = -\frac{3}{5}$ (if we choose 2α in Quadrant II);
thus $\sin\alpha = \sqrt{\frac{1-\cos 2\alpha}{2}} = \sqrt{\frac{1+\left(\frac{3}{5}\right)}{2}} = \frac{2}{\sqrt{5}}$ and $\cos\alpha = \sqrt{\frac{1+\cos 2\alpha}{2}} = \sqrt{\frac{1-\left(\frac{3}{5}\right)}{2}} = \frac{1}{\sqrt{5}}$
$\left(\text{or } \sin\alpha = -\frac{1}{\sqrt{5}} \text{ and } \cos\alpha = \frac{2}{\sqrt{5}}\right)$

29. $\tan 2\alpha = \frac{-1}{1-3} = \frac{1}{2} \Rightarrow 2\alpha \approx 26.57°$
$\Rightarrow \alpha \approx 13.28°$
$\Rightarrow \sin\alpha \approx 0.23, \cos\alpha \approx 0.97$; then $A' \approx 0.88$, $B' \approx 0.00$, $C' \approx 3.12$, $D' \approx 0.74$, $E' \approx -1.20$, and $F' = -3$
$\Rightarrow 0.88x'^2 + 3.12y'^2 + 0.74x' - 1.20y' - 3 = 0$,
an ellipse

30. $\tan 2\alpha = \dfrac{1}{2-(-3)} = \dfrac{1}{5} \Rightarrow 2\alpha \approx 11.31°$
$\Rightarrow \alpha \approx 5.65°$
$\Rightarrow \sin\alpha \approx 0.10, \cos\alpha \approx 0.995$; then $A' \approx 2.05$, $B' \approx 0.00$, $C' \approx -3.05$, $D' \approx 2.99$, $E' \approx -0.30$, and $F' = -7$
$\Rightarrow 2.05x'^2 - 3.05y'^2 + 2.99x' - 0.30y' - 7 = 0$, a hyperbola

31. $\tan 2\alpha = \dfrac{-4}{1-4} = \dfrac{4}{3} \Rightarrow 2\alpha \approx 53.13°$
$\Rightarrow \alpha \approx 26.57°$
$\Rightarrow \sin\alpha \approx 0.45, \cos\alpha \approx 0.89$; then $A' \approx 0.00$, $B' \approx 0.00$, $C' \approx 5.00$, $D' = 0$, $E' = 0$, and $F' = -5$
$\Rightarrow 5.00y'^2 - 5 = 0$ or $y' = \pm 1.00$, parallel lines

32. $\tan 2\alpha = \dfrac{-12}{2-18} = \dfrac{3}{4} \Rightarrow 2\alpha \approx 36.87°$
$\Rightarrow \alpha \approx 18.43°$
$\Rightarrow \sin\alpha \approx 0.32, \cos\alpha \approx 0.95$; then $A' \approx 0.00$, $B' \approx 0.00$, $C' \approx 20.00$, $D' = 0$, $E' = 0$, and $F' = -49$
$\Rightarrow 20.00y'^2 - 49 = 0$, parallel lines

33. $\tan 2\alpha = \dfrac{5}{3-2} = 5 \Rightarrow 2\alpha \approx 78.69°$
$\Rightarrow \alpha \approx 39.35°$
$\Rightarrow \sin\alpha \approx 0.63, \cos\alpha \approx 0.77$; then $A' \approx 5.05$, $B' \approx 0.00$, $C' \approx -0.05$, $D' \approx -5.07$, $E' \approx -6.19$, and $F' = -1$
$\Rightarrow 5.05x'^2 - 0.05y'^2 - 5.07x' - 6.19y' - 1 = 0$, a hyperbola

34. $\tan 2\alpha = \dfrac{7}{2-9} = -1 \Rightarrow 2\alpha = -45°$
$\Rightarrow \alpha = -22.5°$
$\Rightarrow \sin\alpha \approx -0.38, \cos\alpha \approx 0.92$;
then $A' \approx 0.55$, $B' \approx 0.00$, $C' \approx 10.45$, $D' \approx 18.48$, $E' \approx 7.65$, and $F' = -86$
$\Rightarrow 0.55x'^2 + 10.45y'^2 + 18.48x' + 7.65y' - 86 = 0$, an ellipse

35. **(a)** $A' = \cos 45° \sin 45° = \left(\dfrac{\sqrt{2}}{2}\right)\left(\dfrac{\sqrt{2}}{2}\right) = \dfrac{1}{2}$,
$B' = D' = E' = 0$,
$C' = -\sin 45° \cos 45° = -\dfrac{1}{2}$, $F' = -1$
$\Rightarrow \dfrac{1}{2}x'^2 - \dfrac{1}{2}y'^2 = 1 \Rightarrow x'^2 - y'^2 = 2$

(b) $A' = \dfrac{1}{2}$, $C' = -\dfrac{1}{2}$ (see part (a) above), $D' = E' = B' = 0$, $F' = -a$
$\Rightarrow \dfrac{1}{2}x'^2 - \dfrac{1}{2}y'^2 = a \Rightarrow x'^2 - y'^2 = 2a$

36. Yes, the graph is a hyperbola; with $AC < 0$ we have $-4AC > 0$ and $B^2 - 4AC > 0$.

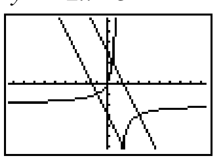
37. The one curve that meets all three of the stated criteria is the ellipse $x^2 + 4xy + 5y^2 - 1 = 0$.
The reasoning: The symmetry about the origin means that $(-x, -y)$ lies on the graph whenever (x, y) does. Adding
$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ and
$A(-x)^2 + B(-x)(-y) + C(-y)^2 + D(-x) + E(-y) + F = 0$
and dividing the result by 2 produces the equivalent equation
$Ax^2 + Bxy + Cy^2 + F = 0$. Substituting $x = 1$, $y = 0$ (because the point (1, 0) lies on the curve) shows further that $A = -F$. Then
$-Fx^2 + Bxy + Cy^2 + F = 0$.
By implicit differentiation,
$-2Fx + By + Bxy' + 2Cyy' = 0$, so substituting $x = -2$, $y = 1$, and $y' = 0$ (from Property 3) gives $4F + B = 0 \Rightarrow B = -4F \Rightarrow$ the conic is
$-Fx^2 - 4Fxy + Cy^2 + F = 0$.
Now substituting $x = -2$ and $y = 1$ again gives $-4F + 8F + C + F = 0 \Rightarrow C = -5F \Rightarrow$ the equation is now $-Fx^2 - 4Fxy - 5Fy^2 + F = 0$.
Finally, dividing through by $-F$ gives the equation $x^2 + 4xy + 5y^2 - 1 = 0$.

38. If $A = C$ then
$B' = B\cos 2\alpha + (C - A)\sin 2\alpha = B\cos 2\alpha$.
Then $\alpha = \dfrac{\pi}{4} \Rightarrow 2\alpha \Rightarrow \dfrac{\pi}{2} \Rightarrow B' = B\cos\dfrac{\pi}{2} = 0$
so the xy-term is eliminated.

39. **(a)** $B^2 - 4AC = 1 - 4(0)(0) = 1 \Rightarrow$ hyperbola

(b) $xy + 2x - y = 0 \Rightarrow y(x-1) = -2x$
$\Rightarrow y = -\dfrac{2x}{x-1}$

(c) $y = -\dfrac{2x}{x-1} \Rightarrow \dfrac{dy}{dx} = \dfrac{2}{(x-1)^2}$ and we want
$\dfrac{-1}{\frac{dy}{dx}} = -2$, the slope of $y = -2x$
$\Rightarrow -2 = -\dfrac{(x-1)^2}{2} \Rightarrow (x-1)^2 = 4$
$\Rightarrow x = 3$ or $x = -1$; $x = 3 \Rightarrow y = -3$
$\Rightarrow (3, -3)$ is a point on the hyperbola where the line with slope $m = -2$ is normal $\Rightarrow$ the line is $y + 3 = -2(x-3)$ or $y = -2x + 3$; $x = -1 \Rightarrow y = -1$
$\Rightarrow (-1, -1)$
is a point on the hyperbola where the line with slope $m = -2$ is normal
$\Rightarrow$ the line is $y + 1 = -2(x+1)$ or
$y = -2x - 3$



[–9.4, 9.4] by [–6.2, 6.2]

40. **(a)** False: let $A = C = 1$, $B = 2$
$\Rightarrow B^2 - 4AC = 0$
$\Rightarrow$ parabola

(b) False: see part (a) above

(c) True:
$AC < 0 \Rightarrow -4AC > 0 \Rightarrow B^2 - 4AC > 0$
$\Rightarrow$ hyperbola

41. $\alpha = 90° \Rightarrow x = x' \cos 90° - y' \sin 90° = -y'$ and $y = x' \sin 90° + y' \cos 90° = x'$

(a) $\dfrac{x'^2}{b^2} + \dfrac{y'^2}{a^2} = 1$

(b) $\dfrac{y'^2}{a^2} - \dfrac{x'^2}{b^2} = 1$

(c) $x'^2 + y'^2 = a^2$

(d) $y = mx \Rightarrow x' = m(-y') \Rightarrow y' = -\dfrac{1}{m}x'$

(e) $y = mx + b \Rightarrow x' = m(-y') + b$
$\Rightarrow y' = -\dfrac{1}{m}x' + \dfrac{b}{m}$

42. $\alpha = 180°$
$\Rightarrow x = x' \cos 180° - y' \sin 180° = -x'$
and $y = x' \sin 180° + y' \cos 180° = -y'$

(a) $\dfrac{x'^2}{a^2} + \dfrac{y'^2}{b^2} = 1$

(b) $\dfrac{x'^2}{a^2} - \dfrac{y'^2}{b^2} = 1$

(c) $x'^2 + y'^2 = a^2$

(d) $y = mx \Rightarrow -y' = m(-x') \Rightarrow y' = mx'$

(e) $y = mx + b \Rightarrow -y' = m(-x') + b$
$\Rightarrow y' = mx' - b$

43. **(a)** $B^2 - 4AC = 4^2 - 4(1)(4) = 0$, so the discriminant indicates that this conic is a parabola.

(b) The left-hand side of
$x^2 + 4xy + 4y^2 + 6x + 12y + 9 = 0$ factors as a perfect square:
$(x + 2y + 3)^2 = 0 \Rightarrow x + 2y + 3 = 0$
$\Rightarrow 2y = -x - 3$;
thus the curve is a degenerate parabola (i.e., a straight line).

44. **(a)** $B^2 - 4AC = 6^2 - 4(9)(1) = 0$, so the discriminant indicates that this conic is a parabola.

(b) The left-hand side of
$9x^2 + 6xy + y^2 - 12x - 4y + 4 = 0$ factors as a perfect square:
$(3x + y - 2)^2 = 0 \Rightarrow 3x + y - 2 = 0$
$\Rightarrow y = -3x + 2$; thus the curve is a degenerate parabola (i.e., a straight line).

45. Assume the ellipse has been rotated to eliminate the xy-term

$\Rightarrow$ the new equation is $A'x'^2 + C'y'^2 = 1 \Rightarrow$ the semi-axes are $\sqrt{\frac{1}{A'}}$ and $\sqrt{\frac{1}{C'}}$

$\Rightarrow$ the area is $\pi\left(\sqrt{\frac{1}{A'}}\right)\left(\sqrt{\frac{1}{C'}}\right) = \frac{\pi}{\sqrt{A'C'}} = \frac{2\pi}{\sqrt{4A'C'}}$

Since $B^2 - 4AC = B'^2 - 4A'C' = -4A'C'$ (because $B' = 0$) we find that the area is $\frac{2\pi}{\sqrt{4AC - B^2}}$ as claimed.

46. **(a)** $A' + C' = (A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha) + (A\sin^2\alpha - B\cos\alpha\sin\alpha + C\cos^2\alpha)$

$$= A(\cos^2\alpha + \sin^2\alpha) + C(\sin^2\alpha + \cos^2\alpha)$$
$$= A + C$$

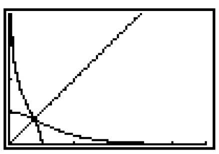
(b) $D'^2 + E'^2 = (D\cos\alpha + E\sin\alpha)^2 + (-D\sin\alpha + E\cos\alpha)^2$

$$= D^2\cos^2\alpha + 2DE\cos\alpha\sin\alpha + E^2\sin^2\alpha + D^2\sin^2\alpha - 2DE\sin\alpha\cos\alpha + E^2\cos^2\alpha$$
$$= D^2(\cos^2\alpha + \sin^2\alpha) + E^2(\sin^2\alpha + \cos^2\alpha)$$
$$= D^2 + E^2$$

Appendix A6 (pp. 611-619)

Exploration 1 Viewing Inverses

1.



(x_1, y_1) forms a line that tends towards infinity.
(x_2, y_2) has a maximum y value of one and tends towards $y = 0$ while x extends towards infinity.
(x_3, y_3) has a maximum x value of one and tends towards $x = 0$ while y extends towards infinity.

2. (x_3, y_3) is the inverse of (x_4, y_4). This means that their graphs will plot in opposite directions.

Exercises

1. $\sinh x = -\frac{3}{4}$

$$\Rightarrow \cosh x = \sqrt{1 + \sinh^2 x}$$
$$= \sqrt{1 + \left(-\frac{3}{4}\right)^2}$$
$$= \sqrt{1 + \frac{9}{16}}$$
$$= \sqrt{\frac{25}{16}}$$
$$= \frac{5}{4},$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(-\frac{3}{4}\right)}{\left(\frac{5}{4}\right)} = -\frac{3}{5},$$

$$\cosh x = \frac{1}{\tanh x} = -\frac{5}{3},$$

$$\text{sech } x = \frac{1}{\cosh x} = \frac{4}{5}, \text{ and}$$

$$\text{csch } x = \frac{1}{\sin x} = -\frac{4}{3}$$

2. $\sinh x = \frac{4}{3}$

$$\Rightarrow \cosh x = \sqrt{1+\sinh^2 x} = \sqrt{1+\left(\frac{4}{3}\right)^2} = \sqrt{1+\frac{16}{9}} = \sqrt{\frac{25}{9}} = \frac{5}{3},$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{4}{3}\right)}{\left(\frac{5}{3}\right)} = \frac{4}{5},$$

$$\coth x = \frac{1}{\tanh x} = \frac{5}{4}, \text{ sech } x = \frac{1}{\cosh x} = \frac{3}{5},$$

$$\text{and csch } x = \frac{1}{\sinh x} = \frac{3}{4}$$

3. $\cosh x = \frac{17}{15},\ x > 0$

$$\Rightarrow \sinh x = \sqrt{\cosh^2 x - 1} = \sqrt{\left(\frac{17}{15}\right)^2 - 1} = \sqrt{\frac{289}{225} - 1} = \sqrt{\frac{64}{225}} = \frac{8}{15},$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{8}{15}\right)}{\left(\frac{17}{15}\right)} = \frac{8}{17},$$

$$\coth x = \frac{1}{\tanh x} = \frac{17}{8},$$

$$\text{sech } x = \frac{1}{\cosh x} = \frac{15}{17}, \text{ and}$$

$$\text{csch } x = \frac{1}{\sinh x} = \frac{15}{8}$$

4. $\cosh x = \frac{13}{5},\ x > 0$

$$\Rightarrow \sinh x = \sqrt{\cosh^2 x - 1} = \sqrt{\frac{169}{25} - 1} = \sqrt{\frac{144}{25}} = \frac{12}{5},$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{12}{5}\right)}{\left(\frac{13}{5}\right)} = \frac{12}{13},$$

$$\coth x = \frac{1}{\tanh x} = \frac{13}{12},\ \text{sech } x = \frac{1}{\cosh x} = \frac{5}{13},$$

$$\text{and csch } x = \frac{1}{\sinh x} = \frac{5}{12}$$

In Exercises 5–10, graphical support may consist of showing that the graph of the original expression minus the simplified one is the line $y = 0$.

5. $$2\cosh(\ln x) = 2\left(\frac{e^{\ln x} + e^{-\ln x}}{2}\right) = e^{\ln x} + \frac{1}{e^{\ln x}} = x + \frac{1}{x}$$

6. $$\sinh(2\ln x) = \frac{e^{2\ln x} - e^{-2\ln x}}{2} = \frac{e^{\ln x^2} - e^{\ln x^{-2}}}{2} = \frac{\left(x^2 - \frac{1}{x^2}\right)}{2} = \frac{x^4 - 1}{2x^2}$$

7. $$\cosh 5x+\sinh 5x=\frac{e^{5x}+e^{-5x}}{2}+\frac{e^{5x}-e^{-5x}}{2}$$
$$=e^{5x}$$

8. $$\cosh 3x-\sinh 3x=\frac{e^{3x}+e^{-3x}}{2}-\frac{e^{3x}-e^{-3x}}{2}$$
$$=e^{-3x}$$

9. $$(\sinh x+\cosh x)^4=\left(\frac{e^{x}-e^{-x}}{2}+\frac{e^{x}+e^{-x}}{2}\right)^4$$
$$=(e^x)^4$$
$$=e^{4x}$$

10. $$\ln(\cosh x+\sinh x)+\ln(\cosh x-\sinh x)=\ln(\cosh^2 x-\sinh^2 x)$$
$$=\ln 1$$
$$=0$$

11. **(a)** $$\sinh 2x=\sinh(x+x)$$
$$=\sinh x\cosh x+\cosh x\sinh x$$
$$=2\sinh x\cosh x$$

(b) $$\cosh 2x=\cosh(x+x)$$
$$=\cosh x\cosh x+\sinh x\sinh x$$
$$=\cosh^2 x+\sinh^2 x$$

12. $$\cosh^2 x-\sinh^2 x=\left(\frac{e^x+e^{-x}}{2}\right)^2-\left(\frac{e^x-e^{-x}}{2}\right)^2=\frac{1}{4}\left[(e^x+e^{-x})+(e^x-e^{-x})\right]\left[(e^x+e^{-x})-(e^x-e^{-x})\right]$$
$$=\frac{1}{4}(2e^x)(2e^{-x})$$
$$=\frac{1}{4}(4e^0)$$
$$=\frac{1}{4}(4)$$
$$=1$$

13. $$y=6\sinh\frac{x}{3}\Rightarrow\frac{dy}{dx}=6\left(\cosh\frac{x}{3}\right)\left(\frac{1}{3}\right)=2\cosh\frac{x}{3}$$

14. $$y=\frac{1}{2}\sinh(2x+1)\Rightarrow\frac{dy}{dx}=\frac{1}{2}\left[\cosh(2x+1)\right](2)=\cosh(2x+1)$$

15. $$y=2\sqrt{t}\tanh\sqrt{t}=2t^{1/2}\tanh t^{1/2}\Rightarrow\frac{dy}{dt}=[\operatorname{sech}^2(t^{1/2})]\left(\frac{1}{2}t^{-1/2}\right)(2t^{1/2})+(\tanh t^{1/2})(t^{-1/2})$$
$$=\operatorname{sech}^2\sqrt{t}+\frac{\tanh\sqrt{t}}{\sqrt{t}}$$

16. $$y=t^2\tanh\frac{1}{t}=t^2\tanh t^{-1}\Rightarrow\frac{dy}{dt}=[\operatorname{sech}^2(t^{-1})](-t^{-2})(t^2)+(2t)(\tanh t^{-1})=-\operatorname{sech}^2\frac{1}{t}+2t\tanh\frac{1}{t}$$

17. $y = \ln(\sinh z) \Rightarrow \frac{dy}{dz} = \frac{\cosh z}{\sinh z} = \coth z$

18. $y = \ln(\cosh z) \Rightarrow \frac{dy}{dz} = \frac{\sinh z}{\cosh z} = \tanh z$

19. $y = (\operatorname{sech}\theta)(1 - \ln \operatorname{sech}\theta) \Rightarrow \frac{dy}{d\theta} = \left(-\frac{-\operatorname{sech}\theta \tanh\theta}{\operatorname{sech}\theta}\right)(\operatorname{sech}\theta) + (-\operatorname{sech}\theta \tanh\theta)(1 - \ln \operatorname{sech}\theta)$

$$\begin{aligned}
&= \operatorname{sech}\theta \tanh\theta - (\operatorname{sech}\theta \tanh\theta)(1 - \ln \operatorname{sech}\theta) \\
&= (\operatorname{sech}\theta \tanh\theta)[1 - (1 - \ln \operatorname{sech}\theta)] \\
&= (\operatorname{sech}\theta \tanh\theta)(\ln \operatorname{sech}\theta)
\end{aligned}$$

20. $y = (\operatorname{csch}\theta)(1 - \ln \operatorname{csch}\theta) \Rightarrow \frac{dy}{d\theta} = (\operatorname{csch}\theta)\left(-\frac{-\operatorname{csch}\theta \coth\theta}{\operatorname{csch}\theta}\right) + (1 - \ln \operatorname{csch}\theta)(-\operatorname{csch}\theta \coth\theta)$

$$\begin{aligned}
&= \operatorname{csch}\theta \coth\theta - (1 - \ln \operatorname{csch}\theta)(\operatorname{csch}\theta \coth\theta) \\
&= (\operatorname{csch}\theta \coth\theta)(1 - 1 + \ln \operatorname{csch}\theta) \\
&= (\operatorname{csch}\theta \coth\theta)(\ln \operatorname{csch}\theta)
\end{aligned}$$

21. $y = \ln \cosh x - \frac{1}{2}\tanh^2 x \Rightarrow \frac{dy}{dx} = \frac{\sinh x}{\cosh x} - \left(\frac{1}{2}\right)(2\tanh x)(\operatorname{sech}^2 x)$

$$\begin{aligned}
&= \tanh x - (\tanh x)(\operatorname{sech}^2 x) \\
&= (\tanh x)(1 - \operatorname{sech}^2 x) \\
&= (\tanh x)(\tanh^2 x) \\
&= \tanh^3 x
\end{aligned}$$

22. $y = \ln \sinh x - \frac{1}{2}\coth^2 x \Rightarrow \frac{dy}{dx} = \frac{\cosh x}{\sinh x} - \left(\frac{1}{2}\right)(2\coth x)(-\operatorname{csch}^2 x)$

$$\begin{aligned}
&= \coth x + (\coth x)(\operatorname{csch}^2 x) \\
&= (\coth x)(1 + \operatorname{csch}^2 x) \\
&= (\coth x)(\coth^2 x) \\
&= \coth^3 x
\end{aligned}$$

23. $y = (x^2 + 1)\operatorname{sech}(\ln x)$

$$\begin{aligned}
&= (x^2 + 1)\left(\frac{2}{e^{\ln x} + e^{-\ln x}}\right) \\
&= (x^2 + 1)\left(\frac{2}{x + x^{-1}}\right) \\
&= (x^2 + 1)\left(\frac{2x}{x^2 + 1}\right) \\
&= 2x \Rightarrow \frac{dy}{dx} = 2
\end{aligned}$$

24. $y = (4x^2 - 1)\,\text{csch}\,(\ln 2x)$
$$= (4x^2-1)\left(\frac{2}{e^{\ln 2x} - e^{-\ln 2x}}\right)$$
$$= (4x^2-1)\left(\frac{2}{2x-(2x)^{-1}}\right)$$
$$= (4x^2-1)\left(\frac{4x}{4x^2-1}\right)$$
$$= 4x \Rightarrow \frac{dy}{dx} = 4$$

25. $y = \sinh^{-1}\sqrt{x} = \sinh^{-1}(x^{1/2})$
$$\Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{2}\right)x^{-1/2}}{\sqrt{1+(x^{1/2})^2}}$$
$$= \frac{1}{2\sqrt{x}\sqrt{1+x}}$$
$$= \frac{1}{2\sqrt{x(1+x)}}$$

26. $y = \cosh^{-1}\left(2\sqrt{x+1}\right) = \cosh^{-1}(2(x+1)^{1/2})$
$$\Rightarrow \frac{dy}{dx} = \frac{(2)\left(\frac{1}{2}\right)(x+1)^{-1/2}}{\sqrt{\left[2(x+1)^{1/2}\right]^2 - 1}}$$
$$= \frac{1}{\sqrt{x+1}\sqrt{4x+3}}$$
$$= \frac{1}{\sqrt{4x^2+7x+3}}$$

27. $y = (1-\theta)\tanh^{-1}\theta$
$$\Rightarrow \frac{dy}{d\theta} = (1-\theta)\left(\frac{1}{1-\theta^2}\right) + (-1)\tanh^{-1}\theta$$
$$= \frac{1}{1+\theta} - \tanh^{-1}\theta$$

28. $y = (\theta^2 + 2\theta)\tanh^{-1}(\theta+1) \Rightarrow \frac{dy}{d\theta} = (\theta^2+2\theta)\left[\frac{1}{1-(\theta+1)^2}\right] + (2\theta+2)\tanh^{-1}(\theta+1)$
$$= \frac{\theta^2+2\theta}{-\theta^2-2\theta} + (2\theta+2)\tanh^{-1}(\theta+1)$$
$$= (2\theta+2)\tanh^{-1}(\theta+1) - 1$$

29. $y = (1-t)\coth^{-1}\sqrt{t} = (1-t)\coth^{-1}(t^{1/2}) \Rightarrow \frac{dy}{dt} = (1-t)\left[\frac{\left(\frac{1}{2}\right)t^{-1/2}}{1-(t^{1/2})^2}\right] + (-1)\coth^{-1}(t^{1/2})$
$$= \frac{1}{2\sqrt{t}} - \coth^{-1}\sqrt{t}$$

30. $y=(1-t^2)\coth^{-1}t \Rightarrow \dfrac{dy}{dt}=(1-t^2)\left(\dfrac{1}{1-t^2}\right)+(-2t)\coth^{-1}t$

$$=1-2t\coth^{-1}t$$

31. $y=\cos^{-1}x-x\,\text{sech}^{-1}x \Rightarrow \dfrac{dy}{dx}=\dfrac{-1}{\sqrt{1-x^2}}-\left[x\left(\dfrac{-1}{x\sqrt{1-x^2}}\right)+(1)\,\text{sech}^{-1}x\right]$

$$=\frac{-1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-x^2}}-\text{sech}^{-1}x$$

$$=-\text{sech}^{-1}x$$

32. $y=\ln x+\sqrt{1-x^2}\;\text{sech}^{-1}x=\ln x+(1-x^2)^{1/2}\,\text{sech}^{-1}x$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{x}+(1-x^2)^{1/2}\left(\frac{-1}{x\sqrt{1-x^2}}\right)+\left(\frac{1}{2}\right)(1-x^2)^{-1/2}(-2x)\;\text{sech}^{-1}x$$

$$=\frac{1}{x}-\frac{1}{x}-\frac{x}{\sqrt{1-x^2}}\;\text{sech}^{-1}x$$

$$=-\frac{x}{\sqrt{1-x^2}}\;\text{sech}^{-1}x$$

33. $y=\text{csch}^{-1}\left(\dfrac{1}{2}\right)^{\theta} \Rightarrow \dfrac{dy}{d\theta}=-\dfrac{\left[\ln\left(\frac{1}{2}\right)\right]\left(\frac{1}{2}\right)^{\theta}}{\left(\frac{1}{2}\right)^{\theta}\sqrt{1+\left[\left(\frac{1}{2}\right)^{\theta}\right]^2}}$

$$=-\frac{\ln(1)-\ln(2)}{\sqrt{1+\left(\frac{1}{2}\right)^{2\theta}}}$$

$$=\frac{\ln 2}{\sqrt{1+\left(\frac{1}{2}\right)^{2\theta}}}$$

34. $y=\text{csch}^{-1}\,2^{\theta} \Rightarrow \dfrac{dy}{d\theta}=-\dfrac{(\ln 2)2^{\theta}}{2^{\theta}\sqrt{1+(2^{\theta})^2}}$

$$=-\frac{\ln 2}{\sqrt{1+2^{2\theta}}}$$

35. $y = \sinh^{-1}(\tan x)$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2 x}{\sqrt{1+(\tan x)^2}} = \frac{\sec^2 x}{\sqrt{\sec^2 x}} = \frac{\sec^2 x}{|\sec x|} = \frac{|\sec x||\sec x|}{|\sec x|} = |\sec x|$$

36. $y = \cosh^{-1}(\sec x)$

$$\Rightarrow \frac{dy}{dx} = \frac{(\sec x)(\tan x)}{\sqrt{\sec^2 x - 1}} = \frac{(\sec x)(\tan x)}{\sqrt{\tan^2 x}} = \frac{(\sec x)(\tan x)}{|\tan x|} = \sec x,\ 0 < x < \frac{\pi}{2}$$

37. **(a)** If $y = \tan^{-1}(\sinh x) + C$, then

$$\frac{dy}{dx} = \frac{\cosh x}{1+\sinh^2 x} = \frac{\cosh x}{\cosh^2 x} = \text{sech } x,$$

which verifies the formula.

(b) If $y = \sin^{-1}(\tanh x) + C$, then

$$\frac{dy}{dx} = \frac{\text{sech}^2 x}{\sqrt{1-\tanh^2 x}} = \frac{\text{sech}^2 x}{\text{sech } x} = \text{sech } x,$$

which verifies the formula.

38. If $y = \frac{x^2}{2}\text{sech}^{-1} x - \frac{1}{2}\sqrt{1-x^2} + C$, then

$$\frac{dy}{dx} = x\,\text{sech}^{-1} x + \frac{x^2}{2}\left(\frac{-1}{x\sqrt{1-x^2}}\right) + \frac{2x}{4\sqrt{1-x^2}}$$

$= x\,\text{sech}^{-1} x$, which verifies the formula.

39. If $y = \frac{x^2-1}{2}\coth^{-1} x + \frac{x}{2} + C$, then

$$\frac{dy}{dx} = x\coth^{-1} x + \left(\frac{x^2-1}{2}\right)\left(\frac{1}{1-x^2}\right) + \frac{1}{2} = x\coth^{-1} x,$$

which verifies the formula.

40. If $y = x\tanh^{-1} x + \frac{1}{2}\ln(1-x^2) + C$,

then $\frac{dy}{dx} = \tanh^{-1} x + x\left(\frac{1}{1-x^2}\right) + \frac{1}{2}\left(\frac{-2x}{1-x^2}\right)$

$= \tanh^{-1} x$,

which verifies the formula.

41. Let $u = 2x$ and $du = 2\,dx$.

$$\int \sinh 2x\,dx = \frac{1}{2}\int \sinh u\,du = \frac{\cosh u}{2} + C = \frac{\cosh 2x}{2} + C$$

42. Let $u = \frac{x}{5}$ and $du = \frac{1}{5}dx$.

$$\int \sinh\frac{x}{5}\,dx = 5\int \sinh u\,du = 5\cosh u + C = 5\cosh\frac{x}{5} + C$$

43. Let $u = \frac{x}{2} - \ln 3$ and $du = \frac{1}{2}dx$.

$$\int 6\cosh\left(\frac{x}{2} - \ln 3\right)dx = 12\int \cosh u\,du = 12\sinh u + C = 12\sinh\left(\frac{x}{2} - \ln 3\right) + C$$

44. Let $u = 3x - \ln 2$ and $du = 3\,dx$.

$$\int 4\cosh(3x - \ln 2)dx = \frac{4}{3}\int \cosh u\,du = \frac{4}{3}\sinh u + C = \frac{4}{3}\sinh(3x - \ln 2) + C$$

45. Let $u = \frac{x}{7}$ and $du = \frac{1}{7}dx$.

$$\int \tanh\frac{x}{7}dx = 7\int \frac{\sinh u}{\cosh u}du = 7\ln|\cosh u| + C = 7\ln|\cosh u| + C_1 = 7\ln\left|\cosh\frac{x}{7}\right| + C$$

46. Let $u = \frac{\theta}{\sqrt{3}}$ and $du = \frac{d\theta}{\sqrt{3}}$.

$$\int \coth \frac{\theta}{\sqrt{3}}\, d\theta = \sqrt{3}\int \frac{\cosh u}{\sinh u}\, du$$
$$= \sqrt{3}\ln|\sinh u| + C$$
$$= \sqrt{3}\ln\left|\sinh \frac{\theta}{\sqrt{3}}\right| + C$$

47. Let $u = \left(x - \frac{1}{2}\right)$ and $du = dx$.

$$\int \operatorname{sech}^2\left(x - \frac{1}{2}\right) dx = \int \operatorname{sech}^2 u\, du = \tanh u + C$$
$$= \tanh\left(x - \frac{1}{2}\right) + C$$

48. Let $u = (5 - x)$ and $du = -dx$.

$$\int \operatorname{csch}^2(5 - x)\, dx = -\int \operatorname{csch}^2 u\, du$$
$$= -(-\coth u) + C$$
$$= \coth u + C$$
$$= \coth(5 - x) + C$$

49. Let $u = \sqrt{t} = t^{1/2}$ and $du = \frac{dt}{2\sqrt{t}}$.

$$\int \frac{\operatorname{sech}\sqrt{t}\tanh\sqrt{t}}{\sqrt{t}}\, dt = 2\int \operatorname{sech} u \tanh u\, du$$
$$= 2(-\operatorname{sech} u) + C$$
$$= -2\operatorname{sech}\sqrt{t} + C$$

50. Let $u = \ln t$ and $du = \frac{dt}{t}$.

$$\int \frac{\operatorname{csch}(\ln t)\coth(\ln t)}{t}\, dt$$
$$= \int \operatorname{csch} u \coth u\, du$$
$$= -\operatorname{csch} u + C$$
$$= -\operatorname{csch}(\ln t) + C$$

51. Let $u = \sinh x$, $du = \cosh x\, dx$, the lower limit is $\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - \left(\frac{1}{2}\right)}{2} = \frac{3}{4}$ and the upper limit is

$$\sinh(\ln 4) = \frac{e^{\ln 4} - e^{-\ln 4}}{2} = \frac{4 - \left(\frac{1}{4}\right)}{2} = \frac{15}{8}.$$

$$\int_{\ln 2}^{\ln 4} \coth x\, dx = \int_{\ln 2}^{\ln 4} \frac{\cosh x}{\sinh x}\, dx$$
$$= \int_{3/4}^{15/8} \frac{1}{u}\, du$$
$$= \Big[\ln|u|\Big]_{3/4}^{15/8}$$
$$= \ln\left|\frac{15}{8}\right| - \ln\left|\frac{3}{4}\right| = \ln\left|\frac{15}{8}\cdot\frac{4}{3}\right|$$
$$= \ln\frac{5}{2} \approx 0.916$$

52. Let $u = \cosh 2x$, $du = 2\sinh(2x)\, dx$, the lower limit is $\cosh 0 = 1$ and the upper limit is

$$\cosh(2\ln 2) = \cosh(\ln 4)$$
$$= \frac{e^{\ln 4} + e^{-\ln 4}}{2}$$
$$= \frac{4 + \left(\frac{1}{4}\right)}{2}$$
$$= \frac{17}{8}$$

$$\int_0^{\ln 2} \tanh 2x\, dx = \int_0^{\ln 2} \frac{\sinh 2x}{\cosh 2x}\, dx$$
$$= \frac{1}{2}\int_1^{17/8} \frac{1}{u}\, du$$
$$= \frac{1}{2}\Big[\ln|u|\Big]_1^{17/8}$$
$$= \frac{1}{2}\left[\ln\left(\frac{17}{8}\right) - \ln 1\right]$$
$$= \frac{1}{2}\ln\frac{17}{8} \approx 0.377$$

53. $\int_{-\ln 4}^{-\ln 2} 2e^{\theta} \cosh\theta \, d\theta$

$= \int_{-\ln 4}^{-\ln 2} 2e^{\theta}\left(\frac{e^{\theta}+e^{-\theta}}{2}\right) d\theta$

$= \int_{-\ln 4}^{-\ln 2} (e^{2\theta}+1)\, d\theta$

$= \left[\frac{e^{2\theta}}{2}+\theta\right]_{-\ln 4}^{-\ln 2}$

$= \left(\frac{e^{-2\ln 2}}{2}-\ln 2\right)-\left(\frac{e^{-2\ln 4}}{2}-\ln 4\right)$

$= \left(\frac{1}{8}-\ln 2\right)-\left(\frac{1}{32}-\ln 4\right)$

$= \frac{3}{32}-\ln 2+2\ \ln 2$

$= \frac{3}{32}+\ln 2$

≈ 0.787

54. $\int_0^{\ln 2} 4e^{-\theta}\sinh\theta\, d\theta$

$= \int_0^{\ln 2} 4e^{-\theta}\left(\frac{e^{\theta}-e^{-\theta}}{2}\right) d\theta$

$= 2\int_0^{\ln 2}(1-e^{-2\theta})\, d\theta$

$= 2\left[\theta+\frac{e^{-2\theta}}{2}\right]_0^{\ln 2}$

$= 2\left[\left(\ln\ 2+\frac{e^{-2\ln 2}}{2}\right)-\left(0+\frac{e^{-0}}{2}\right)\right]$

$= 2\left(\ln\ 2+\frac{1}{8}-\frac{1}{2}\right)$

$= 2\ \ln 2+\frac{1}{4}-1$

$= \ln 4-\frac{3}{4}\approx 0.636$

55. $\int_{-\pi/4}^{\pi/4}\cosh(\tan\theta)\ \sec^2\theta\, d\theta$

$= \int_{-1}^{1}\cosh u\ du$

$= [\sinh\ u]_{-1}^{1} = \sinh\ (1)-\sinh\ (-1)$

$= \left(\frac{e^1-e^{-1}}{2}\right)-\left(\frac{e^{-1}-e^1}{2}\right)$

$= \frac{e-e^{-1}-e^{-1}+e}{2}$

$= e-e^{-1}\approx 2.350,$ where $u=\tan\ \theta,$ $du=\sec^2\theta\, d\theta,$ the lower limit is $\tan\left(-\frac{\pi}{4}\right)=-1$ and the upper limit is $\tan\left(\frac{\pi}{4}\right)=1.$

56. $\int_0^{\pi/2} 2\ \sinh(\sin\theta)\cos\ \theta\, d\theta$

$= 2\int_0^1 \sinh u\ du$

$= 2[\cosh\ u]_0^1$

$= 2(\cosh 1-\cosh 0)$

$= 2\left(\frac{e+e^{-1}}{2}-1\right)$

$= e+e^{-1}-2\approx 1.086,$

where $u=\sin\theta,$ $du=\cos\theta\, d\theta,$ the lower limit is $\sin 0 = 0$ and the upper limit is $\sin\left(\frac{\pi}{2}\right)=1.$

57. $\int_1^2 \frac{\cosh\ (\ln t)}{t}dt = \int_0^{\ln 2}\cosh\ u\ du$

$= [\sinh u]_0^{\ln 2}$

$= \sinh(\ln 2)-\sinh(0)$

$= \frac{e^{\ln 2}-e^{-\ln 2}}{2}-0$

$= \frac{2-\frac{1}{2}}{2}$

$= \frac{3}{4},$

where $u=\ln t,$ $du=\frac{1}{t}dt,$ the lower limit is $\ln 1 = 0$ and the upper limit is $\ln 2$.

58. $\int_1^4 \frac{8\cosh\sqrt{x}}{\sqrt{x}}dx$

$= 16\int_1^2 \cosh\ u\ du$

$= 16[\sinh\ u]_1^2$

$= 16(\sinh 2-\sinh 1)$

$= 16\left[\left(\frac{e^2-e^{-2}}{2}\right)-\left(\frac{e-e^{-1}}{2}\right)\right]$

$= 8(e^2-e^{-2}-e+e^{-1})\approx 39.227,$

where $u=\sqrt{x}=x^{1/2},$ $du=\frac{1}{2}x^{-1/2}\ dx=\frac{dx}{2\sqrt{x}},$ the lower limit is $\sqrt{1}=1$ and the upper limit is $\sqrt{4}=2.$

59. $\int_{-\ln 2}^{0} \cosh^2\left(\frac{x}{2}\right) dx = \int_{-\ln 2}^{0} \frac{\cosh x + 1}{2} dx$

$$\begin{aligned}
&= \frac{1}{2}\int_{-\ln 2}^{0} (\cosh x + 1) dx \\
&= \frac{1}{2}\left[\sinh x + x\right]_{-\ln 2}^{0} \\
&= \frac{1}{2}[(\sinh 0 + 0) - (\sinh\ (-\ln 2) - \ln 2)] \\
&= \frac{1}{2}\left[(0+0) - \left(\frac{e^{-\ln 2} - e^{\ln 2}}{2} - \ln 2\right)\right] \\
&= \frac{1}{2}\left[-\frac{\left(\frac{1}{2}\right) - 2}{2} + \ln 2\right] \\
&= \frac{1}{2}\left(1 - \frac{1}{4} + \ln 2\right) \\
&= \frac{3}{8} + \frac{1}{2}\ln 2 \\
&= \frac{3}{8} + \ln\sqrt{2} \approx 0.722
\end{aligned}$$

60. $\int_{0}^{\ln 10} 4\sinh^2\left(\frac{x}{2}\right) dx = \int_{0}^{\ln 10} 4\left(\frac{\cosh x - 1}{2}\right) dx$

$$\begin{aligned}
&= 2\int_{0}^{\ln 10} (\cosh x - 1)\ dx \\
&= 2\left[\sinh x - x\right]_{0}^{\ln 10} \\
&= 2[(\sinh\ (\ln 10) - \ln 10) - (\sinh 0 - 0)] \\
&= e^{\ln 10} - e^{-\ln 10} - 2\ln 10 \\
&= 10 - \frac{1}{10} - 2\ln 10 \\
&= 9.9 - 2\ln 10 \approx 5.295
\end{aligned}$$

61. $\cosh^2 x - \sinh^2 x = 1$, so $\int_0^2 \pi(\cosh^2 x - \sinh^2 x)\ dx = \pi\int_0^2 1\ dx = 2\pi.$

62. $\int_{-\ln\sqrt{3}}^{\ln\sqrt{3}} \pi\,\mathrm{sec\,h}^2 x\ dx = \pi\left[\tanh x\right]_{-\ln\sqrt{3}}^{\ln\sqrt{3}}$

$$\begin{aligned}
&= \pi\left(\frac{e^{\ln\sqrt{3}} - e^{-\ln\sqrt{3}}}{e^{\ln\sqrt{3}} + e^{-\ln\sqrt{3}}} - \frac{e^{-\ln\sqrt{3}} - e^{\ln\sqrt{3}}}{e^{-\ln\sqrt{3}} + e^{\ln\sqrt{3}}}\right) \\
&= 2\pi\left(\frac{e^{\ln\sqrt{3}} - e^{-\ln\sqrt{3}}}{e^{\ln\sqrt{3}} + e^{-\ln\sqrt{3}}}\right) \\
&= 2\pi\left(\frac{\sqrt{3} - \frac{1}{\sqrt{3}}}{\sqrt{3} + \frac{1}{\sqrt{3}}}\right) \\
&= 2\pi\left(\frac{3-1}{3+1}\right) \\
&= \pi
\end{aligned}$$

63. $$\int_0^{\ln\sqrt{199}} \pi(1-\tanh x)^2\,dx = \pi\int_0^{\ln\sqrt{199}} (1-2\tanh x+\tanh^2 x)\ dx$$
$$= \pi\int_0^{\ln\sqrt{199}} (2-2\tanh x-\operatorname{sech}^2 x)\ dx$$
$$= \pi\left[2x-2\ln(\cosh x)-\tanh x\right]_0^{\ln\sqrt{199}}$$
$$= \pi\left[2\ln\sqrt{199}-2\ln\left[\cosh\left(\ln\sqrt{199}\right)\right]-\tanh\left(\ln\sqrt{199}\right)\right]$$
$$= \pi\left[2\ln\sqrt{199}-2\ln\left(\frac{e^{\ln\sqrt{199}}+e^{-\ln\sqrt{199}}}{2}\right)-\frac{e^{\ln\sqrt{199}}-e^{-\ln\sqrt{199}}}{e^{\ln\sqrt{199}}+e^{-\ln\sqrt{199}}}\right]$$
$$= \pi\left[\ln 199-\ln\left(\frac{\left(\sqrt{199}+\frac{1}{\sqrt{199}}\right)^2}{4}\right)-\frac{\sqrt{199}-\frac{1}{\sqrt{199}}}{\sqrt{199}+\frac{1}{\sqrt{199}}}\right]$$
$$= \pi\left[\ln 199-\ln\left(\frac{199+2+\frac{1}{199}}{4}\right)-\frac{199-1}{199+1}\right]$$
$$= \pi\left[\ln 199-\ln\left(\frac{10,000}{199}\right)-\frac{99}{100}\right]$$
$$= \left(2\ln\frac{199}{100}-\frac{99}{100}\right)\pi \approx 1.214$$

64. **(a)** $y=\frac{1}{2}\cosh 2x \Rightarrow y'=\sinh 2x$
$$\Rightarrow L=\int_0^{\ln\sqrt{5}}\sqrt{1+(\sinh 2x)^2}\,dx$$
$$=\int_0^{\ln\sqrt{5}}\cosh\ 2x\ dx$$
$$=\left[\frac{1}{2}\sinh 2x\right]_0^{\ln\sqrt{5}}$$
$$=\left[\frac{1}{2}\left(\frac{e^{2x}-e^{-2x}}{2}\right)\right]_0^{\ln\sqrt{5}}$$
$$=\frac{1}{4}\left(5-\frac{1}{5}\right)$$
$$=\frac{6}{5}$$

(b) $y=\frac{1}{a}\cosh ax \Rightarrow 1+(y')^2=1+\sinh^2 ax=\cosh^2 ax$
$$\Rightarrow L=\int_0^b\sqrt{\cosh^2 ax}\ dx$$
$$=\int_0^b\cosh ax\ dx$$
$$=\left[\frac{\sinh ax}{a}\right]_0^b$$
$$=\frac{\sinh ab}{a}$$

65. **(a)** Let $E(x)=\dfrac{f(x)+f(-x)}{2}$ and $O(x)=\dfrac{f(x)-f(-x)}{2}$. Then

$$E(x)+O(x)=\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2}=\frac{2f(x)}{2}=f(x).$$

Also, $$E(-x)=\frac{f(-x)+f(-(-x))}{2}=\frac{f(x)+f(-x)}{2}=E(x)$$

$\Rightarrow E(x)$ is even, and

$$O(-x)=\frac{f(-x)-f(-(-x))}{2}=-\frac{f(x)-f(-x)}{2}=-O(x)$$

$\Rightarrow O(x)$ is odd. Consequently, $f(x)$ can be written as a sum of an even and an odd function.

(b) Even part: $\dfrac{e^x+e^{-x}}{2}=\cosh x$

odd part: $\dfrac{e^x-e^{-x}}{2}=\sinh x$

66. **(a)** If f is even, then

$$\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2}=\frac{2f(x)}{2}+\frac{f(x)-f(x)}{2}=f(x)+0$$

(b) If f is odd, then

$$\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2}=\frac{f(x)-f(x)}{2}+\frac{f(x)+f(x)}{2}=0+f(x)$$

67. Note that $$\frac{dv}{dt}=\sqrt{\frac{mg}{k}}\operatorname{sech}^2\left(\sqrt{\frac{gk}{m}}t\right)\left(\sqrt{\frac{gk}{m}}\right)=g\operatorname{sech}^2\left(\sqrt{\frac{gk}{m}}t\right).$$

Then $m\dfrac{dv}{dt}=mg\operatorname{sech}^2\left(\sqrt{\dfrac{gk}{m}}t\right)$ and

$$mg-kv^2=mg-k\left[\sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{gk}{m}}t\right)\right]^2=mg\left[1-\tanh^2\left(\sqrt{\frac{gk}{m}}t\right)\right]=mg\operatorname{sech}^2\left(\sqrt{\frac{gk}{m}}t\right).$$

Thus, $m\dfrac{dv}{dt}$ and $mg-kv^2$ are equal to the same quantity, so the differential equation is satisfied. Furthermore, the initial condition is satisfied because $v(0)=\sqrt{\dfrac{mg}{k}}\tanh 0=0.$

68. **(a)** $s(t)=a\cos kt+b\sin kt$

$$\Rightarrow\frac{ds}{dt}=-ak\sin kt+bk\cos kt$$

$$\Rightarrow\frac{d^2s}{dt^2}=-ak^2\cos kt-bk^2\sin kt=-k^2(a\cos kt+\sin kt)=-k^2s(t)$$

$\Rightarrow$ acceleration is proportional to s. The negative constant $-k^2$ implies that the acceleration is directed toward the origin.

(b) $s(t)=a\cosh kt+b\sinh kt$

$$\Rightarrow\frac{ds}{dt}=ak\sinh kt+bk\cosh kt$$

$$\Rightarrow\frac{d^2s}{dt^2}=ak^2\cosh kt+bk^2\sinh kt=k^2(a\cosh kt+\sinh kt)=k^2s(t)$$

$\Rightarrow$ acceleration is proportional to s. The positive constant k^2 implies that the acceleration is directed away from the origin.

69. $\frac{dy}{dx} = \frac{-1}{x\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} \Rightarrow y = \int \frac{-1}{x\sqrt{1-x^2}}\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx$
$\Rightarrow y = \operatorname{sech}^{-1}(x) - \sqrt{1-x^2} + C$; $x = 1$ and $y = 0$
$\Rightarrow C = 0 \Rightarrow y = \operatorname{sech}^{-1}(x) - \sqrt{1-x^2}$

70. $y = 4\cosh\frac{x}{4} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \sinh^2\left(\frac{x}{4}\right) = \cosh^2\left(\frac{x}{4}\right)$; the surface area is

$$\begin{aligned}
S &= \int_{-\ln 16}^{\ln 81} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\
&= 8\pi \int_{-\ln 16}^{\ln 81} \cosh^2\left(\frac{x}{4}\right) dx = 4\pi \int_{-\ln 16}^{\ln 81} \left(1 + \cosh\frac{x}{2}\right) dx \\
&= 4\pi \left[x + 2\sinh\frac{x}{2}\right]_{-\ln 16}^{\ln 81} \\
&= 4\pi \left[\left(\ln 81 + 2\sinh\left(\frac{\ln 81}{2}\right)\right) - \left(-\ln 16 + 2\sinh\left(\frac{-\ln 16}{2}\right)\right)\right] \\
&= 4\pi [\ln(81\cdot 16) + 2\sinh(\ln 9) + 2\sinh(\ln 4)] \\
&= 4\pi [\ln(9\cdot 4)^2 + (e^{\ln 9} - e^{-\ln 9}) + (e^{\ln 4} - e^{-\ln 4})] \\
&= 4\pi \left[2\ln 36 + \left(9 - \frac{1}{9}\right) + \left(4 - \frac{1}{4}\right)\right] \\
&= 4\pi \left(4\ln 6 + \frac{80}{9} + \frac{15}{4}\right) \\
&= 4\pi \left(4\ln 6 + \frac{320+135}{36}\right) \\
&= 16\pi \ln 6 + \frac{455\pi}{9} \approx 248.889
\end{aligned}$$

71. $y = a\cosh\left(\frac{x}{a}\right)$

$$\begin{aligned}
&\Rightarrow y' = \sinh\left(\frac{x}{a}\right) \\
&\Rightarrow y'' = \left(\frac{1}{a}\right)\cosh\left(\frac{x}{a}\right) \\
&\qquad = \left(\frac{1}{a}\right)\sqrt{\cosh^2\left(\frac{x}{a}\right)} \\
&\qquad = \left(\frac{1}{a}\right)\sqrt{1 + \sinh^2\left(\frac{x}{a}\right)} \\
&\qquad = \left(\frac{1}{a}\right)\sqrt{1 + (y')^2}.
\end{aligned}$$

Also, $y'(0) = \sinh(0) = 0$ and $y(0) = a\cosh(0) = a$.

72. **(a)** Let the point located at ($\cosh x$, 0) be called T. Then $A(u)$ = area of the triangle ΔOTP minus the area under the curve $y = \sqrt{x^2 - 1}$ from A to $T \Rightarrow A(u) = \frac{1}{2}\cosh u \sinh u - \int_1^{\cosh u} \sqrt{x^2 - 1}\,dx$.

(b) $A(u) = \frac{1}{2}\cosh u \sinh u - \int_1^{\cosh u} \sqrt{x^2 - 1}\, dx$

$$\Rightarrow A'(u) = \frac{1}{2}(\cosh^2 u + \sinh^2 u) - (\sqrt{\cosh^2 u - 1})(\sinh u)$$
$$= \frac{1}{2}\cosh^2 u + \frac{1}{2}\sinh^2 u - \sinh^2 u$$
$$= \frac{1}{2}(\cosh^2 u - \sinh^2 u)$$
$$= \left(\frac{1}{2}\right)(1)$$
$$= \frac{1}{2}$$

(c) $A'(u) = \frac{1}{2} \Rightarrow A(u) = \frac{u}{2} + C,$ and from part (a) we have $A(0) = 0 \Rightarrow C = 0 \Rightarrow A(u) = \frac{u}{2} \Rightarrow u = 2A(u)$